

Elementary Linear Algebra

8e

Ron Larson

The Pennsylvania State University
The Behrend College



Australia • Brazil • Mexico • Singapore • United Kingdom • United States

Elementary Linear Algebra
Eighth Edition
Ron Larson

Product Director: Terry Boyle
Product Manager: Richard Stratton
Content Developer: Spencer Arritt
Product Assistant: Kathryn Schrupf
Marketing Manager: Ana Albinson
Content Project Manager: Jennifer Riden
Manufacturing Planner: Doug Bertke
Production Service: Larson Texts, Inc.
Photo Researcher: Lumina Datamatics
Text Researcher: Lumina Datamatics
Text Designer: Larson Texts, Inc.
Cover Designer: Larson Texts, Inc.
Cover Image: Keo/Shutterstock.com
Compositor: Larson Texts, Inc.

© 2017, 2013, 2009 Cengage Learning

WCN: 02-200-203

ALL RIGHTS RESERVED. No part of this work covered by the copyright herein may be reproduced, transmitted, stored, or used in any form or by any means graphic, electronic, or mechanical, including but not limited to photocopying, recording, scanning, digitizing, taping, Web distribution, information networks, or information storage and retrieval systems, except as permitted under Section 107 or 108 of the 1976 United States Copyright Act, without the prior written permission of the publisher.

For product information and technology assistance, contact us at
Cengage Learning Customer & Sales Support, 1-800-354-9706.

For permission to use material from this text or product,
submit all requests online at **www.cengage.com/permissions**.
Further permissions questions can be e-mailed to
permissionrequest@cengage.com.

Library of Congress Control Number: 2015944033

Student Edition
ISBN: 978-1-305-65800-4

Loose-leaf Edition
ISBN: 978-1-305-95320-8

Cengage Learning
20 Channel Center Street
Boston, MA 02210
USA

Cengage Learning is a leading provider of customized learning solutions with employees residing in nearly 40 different countries and sales in more than 125 countries around the world. Find your local representative at **www.cengage.com**.

Cengage Learning products are represented in Canada by Nelson Education, Ltd.

To learn more about Cengage Learning Solutions, visit **www.cengage.com**.
Purchase any of our products at your local college store or at our preferred online store **www.cengagebrain.com**.

Contents

1	■	Systems of Linear Equations	1
1.1		Introduction to Systems of Linear Equations	2
1.2		Gaussian Elimination and Gauss-Jordan Elimination	13
1.3		Applications of Systems of Linear Equations	25
		<i>Review Exercises</i>	35
		<i>Project 1 Graphing Linear Equations</i>	38
		<i>Project 2 Underdetermined and Overdetermined Systems</i>	38
2	■	Matrices	39
2.1		Operations with Matrices	40
2.2		Properties of Matrix Operations	52
2.3		The Inverse of a Matrix	62
2.4		Elementary Matrices	74
2.5		Markov Chains	84
2.6		More Applications of Matrix Operations	94
		<i>Review Exercises</i>	104
		<i>Project 1 Exploring Matrix Multiplication</i>	108
		<i>Project 2 Nilpotent Matrices</i>	108
3	■	Determinants	109
3.1		The Determinant of a Matrix	110
3.2		Determinants and Elementary Operations	118
3.3		Properties of Determinants	126
3.4		Applications of Determinants	134
		<i>Review Exercises</i>	144
		<i>Project 1 Stochastic Matrices</i>	147
		<i>Project 2 The Cayley-Hamilton Theorem</i>	147
		<i>Cumulative Test for Chapters 1–3</i>	149
4	■	Vector Spaces	151
4.1		Vectors in R^n	152
4.2		Vector Spaces	161
4.3		Subspaces of Vector Spaces	168
4.4		Spanning Sets and Linear Independence	175
4.5		Basis and Dimension	186
4.6		Rank of a Matrix and Systems of Linear Equations	195
4.7		Coordinates and Change of Basis	208
4.8		Applications of Vector Spaces	218
		<i>Review Exercises</i>	227
		<i>Project 1 Solutions of Linear Systems</i>	230
		<i>Project 2 Direct Sum</i>	230

5	■ Inner Product Spaces	231
5.1	Length and Dot Product in R^n	232
5.2	Inner Product Spaces	243
5.3	Orthonormal Bases: Gram-Schmidt Process	254
5.4	Mathematical Models and Least Squares Analysis	265
5.5	Applications of Inner Product Spaces	277
	<i>Review Exercises</i>	290
	<i>Project 1 The QR-Factorization</i>	293
	<i>Project 2 Orthogonal Matrices and Change of Basis</i>	294
	<i>Cumulative Test for Chapters 4 and 5</i>	295
6	■ Linear Transformations	297
6.1	Introduction to Linear Transformations	298
6.2	The Kernel and Range of a Linear Transformation	309
6.3	Matrices for Linear Transformations	320
6.4	Transition Matrices and Similarity	330
6.5	Applications of Linear Transformations	336
	<i>Review Exercises</i>	343
	<i>Project 1 Reflections in R^2 (I)</i>	346
	<i>Project 2 Reflections in R^2 (II)</i>	346
7	■ Eigenvalues and Eigenvectors	347
7.1	Eigenvalues and Eigenvectors	348
7.2	Diagonalization	359
7.3	Symmetric Matrices and Orthogonal Diagonalization	368
7.4	Applications of Eigenvalues and Eigenvectors	378
	<i>Review Exercises</i>	393
	<i>Project 1 Population Growth and Dynamical Systems (I)</i>	396
	<i>Project 2 The Fibonacci Sequence</i>	396
	<i>Cumulative Test for Chapters 6 and 7</i>	397
8	■ Complex Vector Spaces (online)*	
8.1	Complex Numbers	
8.2	Conjugates and Division of Complex Numbers	
8.3	Polar Form and DeMoivre's Theorem	
8.4	Complex Vector Spaces and Inner Products	
8.5	Unitary and Hermitian Matrices	
	<i>Review Exercises</i>	
	<i>Project 1 The Mandelbrot Set</i>	
	<i>Project 2 Population Growth and Dynamical Systems (II)</i>	

9 **Linear Programming (online)***

- 9.1** Systems of Linear Inequalities
- 9.2** Linear Programming Involving Two Variables
- 9.3** The Simplex Method: Maximization
- 9.4** The Simplex Method: Minimization
- 9.5** The Simplex Method: Mixed Constraints

Review Exercises

Project 1 Beach Sand Replenishment (I)

Project 2 Beach Sand Replenishment (II)

10 **Numerical Methods (online)***

- 10.1** Gaussian Elimination with Partial Pivoting
- 10.2** Iterative Methods for Solving Linear Systems
- 10.3** Power Method for Approximating Eigenvalues
- 10.4** Applications of Numerical Methods

Review Exercises

Project 1 The Successive Over-Relaxation (SOR) Method

Project 2 United States Population

Appendix

A1

Mathematical Induction and Other Forms of Proofs

Answers to Odd-Numbered Exercises and Tests

A7

Index

A41

Technology Guide*

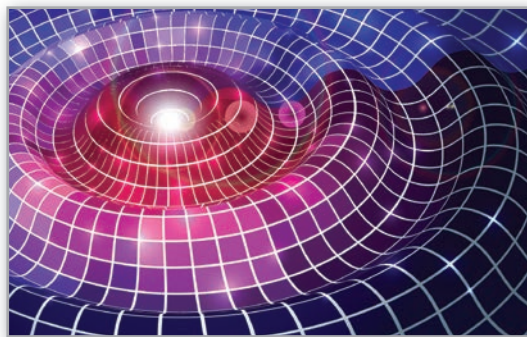
*Available online at **CengageBrain.com**.

7 Eigenvalues and Eigenvectors

- 7.1 Eigenvalues and Eigenvectors
- 7.2 Diagonalization
- 7.3 Symmetric Matrices and Orthogonal Diagonalization
- 7.4 Applications of Eigenvalues and Eigenvectors



Population of Rabbits (p. 379)



Relative Maxima and Minima (p. 375)



Architecture (p. 388)



Genetics (p. 365)



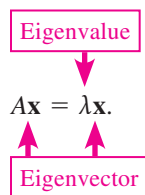
Diffusion (p. 354)

7.1 Eigenvalues and Eigenvectors

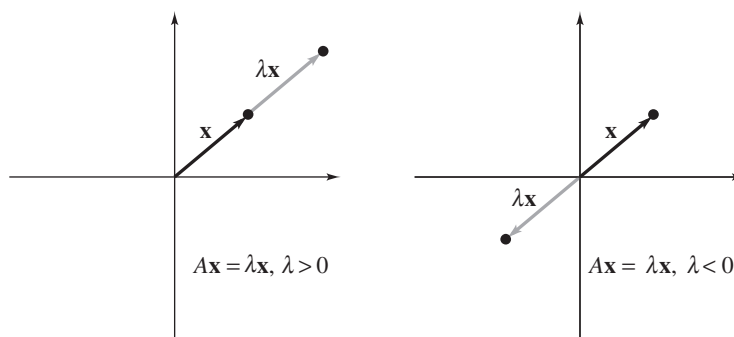
- Verify eigenvalues and corresponding eigenvectors.
- Find eigenvalues and corresponding eigenspaces.
- Use the characteristic equation to find eigenvalues and eigenvectors, and find the eigenvalues and eigenvectors of a triangular matrix.
- Find the eigenvalues and eigenvectors of a linear transformation.

THE EIGENVALUE PROBLEM

This section presents one of the most important problems in linear algebra, the **eigenvalue problem**. Its central question is “when A is an $n \times n$ matrix, do nonzero vectors $\mathbf{x}$ in R^n exist such that $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$?” The scalar, denoted by the Greek letter lambda (λ), is called an **eigenvalue** of the matrix A , and the nonzero vector $\mathbf{x}$ is called an **eigenvector** of A corresponding to λ . The origins of the terms *eigenvalue* and *eigenvector* are from the German word *Eigenwert*, meaning “proper value.” So, you have



Eigenvalues and eigenvectors have many important applications, many of which are discussed throughout this chapter. For now, you will consider a geometric interpretation of the problem in R^2 . If λ is an eigenvalue of a matrix A and $\mathbf{x}$ is an eigenvector of A corresponding to λ , then multiplication of $\mathbf{x}$ by the matrix A produces a vector $\lambda\mathbf{x}$ that is parallel to $\mathbf{x}$, as shown below.



REMARK

Only eigenvectors of real eigenvalues are presented in this chapter.

Definitions of Eigenvalue and Eigenvector

Let A be an $n \times n$ matrix. The scalar λ is an **eigenvalue** of A when there is a *nonzero* vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is an **eigenvector** of A corresponding to λ .

Note that an *eigenvector* cannot be zero. Allowing $\mathbf{x}$ to be the zero vector would render the definition meaningless, because $A\mathbf{0} = \lambda\mathbf{0}$ is true for all real values of λ . An *eigenvalue* of $\lambda = 0$, however, is possible. (See Example 2.)

A matrix can have more than one eigenvalue, as demonstrated in Examples 1 and 2.

EXAMPLE 1**Verifying Eigenvectors and Eigenvalues**

For the matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

verify that $\mathbf{x}_1 = (1, 0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 2$, and that $\mathbf{x}_2 = (0, 1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = -1$.

SOLUTION

Multiplying $\mathbf{x}_1$ on the left by A produces

$$A\mathbf{x}_1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

So, $\mathbf{x}_1 = (1, 0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 2$. Similarly, multiplying $\mathbf{x}_2$ on the left by A produces

$$A\mathbf{x}_2 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

So, $\mathbf{x}_2 = (0, 1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = -1$.

EXAMPLE 2**Verifying Eigenvectors and Eigenvalues**

For the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

verify that

$$\mathbf{x}_1 = (-3, -1, 1) \quad \text{and} \quad \mathbf{x}_2 = (1, 0, 0)$$

are eigenvectors of A and find their corresponding eigenvalues.

SOLUTION

Multiplying $\mathbf{x}_1$ on the left by A produces

$$A\mathbf{x}_1 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}.$$

So, $\mathbf{x}_1 = (-3, -1, 1)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 0$. Similarly, multiplying $\mathbf{x}_2$ on the left by A produces

$$A\mathbf{x}_2 = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

So, $\mathbf{x}_2 = (1, 0, 0)$ is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = 1$.

DISCOVERY

1. In Example 2, $\lambda_2 = 1$ is an eigenvalue of the matrix A . Calculate the determinant of the matrix $\lambda_2 I - A$, where I is the 3×3 identity matrix.
2. Repeat for the other eigenvalue, $\lambda_1 = 0$.
3. In general, when λ is an eigenvalue of the matrix A , what is the value of $|\lambda I - A|$?

EIGENSPACES

Although Examples 1 and 2 list only one eigenvector for each eigenvalue, each of the four eigenvalues in Examples 1 and 2 has infinitely many eigenvectors. For instance, in Example 1, the vectors $(2, 0)$ and $(-3, 0)$ are eigenvectors of A corresponding to the eigenvalue 2. In fact, if A is an $n \times n$ matrix with an eigenvalue λ and a corresponding eigenvector $\mathbf{x}$, then every nonzero scalar multiple of $\mathbf{x}$ is also an eigenvector of A . To see this, let c be a nonzero scalar, which then produces

$$A(c\mathbf{x}) = c(A\mathbf{x}) = c(\lambda\mathbf{x}) = \lambda(c\mathbf{x}).$$

It is also true that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are eigenvectors corresponding to the *same* eigenvalue λ , then their sum is also an eigenvector corresponding to λ , because

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \lambda\mathbf{x}_1 + \lambda\mathbf{x}_2 = \lambda(\mathbf{x}_1 + \mathbf{x}_2).$$

In other words, the set of all eigenvectors of an eigenvalue λ , together with the zero vector, is a subspace of R^n . This special subspace of R^n is called the **eigenspace** of λ .

THEOREM 7.1 Eigenvectors of λ Form a Subspace

If A is an $n \times n$ matrix with an eigenvalue λ , then the set of all eigenvectors of λ , together with the zero vector

$$\{\mathbf{x}: \mathbf{x} \text{ is an eigenvector of } \lambda\} \cup \{\mathbf{0}\}$$

is a subspace of R^n . This subspace is the **eigenspace** of λ .

Determining the eigenvalues and corresponding eigenspaces of a matrix can involve algebraic manipulation. Occasionally, however, it is possible to find eigenvalues and eigenspaces by inspection, as demonstrated in Example 3.

EXAMPLE 3 Finding Eigenspaces in R^2 Geometrically

Find the eigenvalues and corresponding eigenspaces of $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

SOLUTION

Geometrically, multiplying a vector (x, y) in R^2 by the matrix A corresponds to a reflection in the y -axis. That is, if $\mathbf{v} = (x, y)$, then

$$A\mathbf{v} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}.$$

Figure 7.1 illustrates that the only vectors reflected onto scalar multiples of themselves are those lying on either the x -axis or the y -axis.

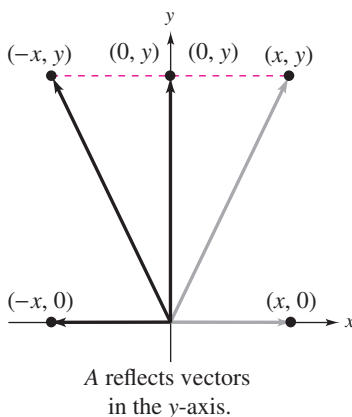


Figure 7.1

For a vector on the x -axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ 0 \end{bmatrix} = -1 \begin{bmatrix} x \\ 0 \end{bmatrix}$$

Eigenvalue is $\lambda_1 = -1$.

For a vector on the y -axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = 1 \begin{bmatrix} 0 \\ y \end{bmatrix}$$

Eigenvalue is $\lambda_2 = 1$.

REMARK

The geometric solution in Example 3 is not typical of the general eigenvalue problem. A more general approach follows.

So, the eigenvectors corresponding to $\lambda_1 = -1$ are the nonzero vectors on the x -axis, and the eigenvectors corresponding to $\lambda_2 = 1$ are the nonzero vectors on the y -axis. This implies that the eigenspace corresponding to $\lambda_1 = -1$ is the x -axis, and that the eigenspace corresponding to $\lambda_2 = 1$ is the y -axis.

FINDING EIGENVALUES AND EIGENVECTORS

To find the eigenvalues and eigenvectors of an $n \times n$ matrix A , let I be the $n \times n$ identity matrix. Rewriting $A\mathbf{x} = \lambda\mathbf{x}$ as $\lambda I\mathbf{x} = A\mathbf{x}$ and rearranging gives $(\lambda I - A)\mathbf{x} = \mathbf{0}$. This homogeneous system of equations has nonzero solutions if and only if the coefficient matrix $(\lambda I - A)$ is *not* invertible—that is, if and only if its determinant is zero. The next theorem formally states this.

REMARK

The characteristic polynomial of A is of degree n , so A can have at most n distinct eigenvalues. The Fundamental Theorem of Algebra states that an n th-degree polynomial has precisely n roots. These n roots, however, include both repeated and complex roots. In this chapter, you will focus on the real roots of characteristic polynomials—that is, real eigenvalues.

THEOREM 7.2 Eigenvalues and Eigenvectors of a Matrix

Let A be an $n \times n$ matrix.

1. An eigenvalue of A is a scalar λ such that $\det(\lambda I - A) = 0$.
2. The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

The equation $\det(\lambda I - A) = 0$ is the **characteristic equation** of A . Moreover, when expanded to polynomial form, the polynomial

$$|\lambda I - A| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_2\lambda^2 + c_1\lambda + c_0$$

is the **characteristic polynomial** of A . So, the eigenvalues of an $n \times n$ matrix A correspond to the roots of the characteristic polynomial of A .

EXAMPLE 4

Finding Eigenvalues and Eigenvectors

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Find the eigenvalues and corresponding eigenvectors of $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$.

SOLUTION

The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} = \lambda^2 + 3\lambda - 10 + 12 = (\lambda + 1)(\lambda + 2).$$

So, the characteristic equation is $(\lambda + 1)(\lambda + 2) = 0$, which gives $\lambda_1 = -1$ and $\lambda_2 = -2$ as the eigenvalues of A . To find the corresponding eigenvectors, solve the homogeneous linear system represented by $(\lambda I - A)\mathbf{x} = \mathbf{0}$ twice: first for $\lambda = \lambda_1 = -1$, and then for $\lambda = \lambda_2 = -2$. For $\lambda_1 = -1$, the coefficient matrix is

$$(-1)I - A = \begin{bmatrix} -1 - 2 & 12 \\ -1 & -1 + 5 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix}$$

which row reduces to $\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$, showing that $x_1 - 4x_2 = 0$. Letting $x_2 = t$, you can conclude that every eigenvector of λ_1 is of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad t \neq 0.$$

For $\lambda_2 = -2$, you have

$$(-2)I - A = \begin{bmatrix} -2 - 2 & 12 \\ -1 & -2 + 5 \end{bmatrix} = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}.$$

Letting $x_2 = t$, you can conclude that every eigenvector of λ_2 is of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad t \neq 0.$$

REMARK

Check that the eigenvalues and eigenvectors in this example satisfy the equation $A\mathbf{x} = \lambda\mathbf{x}$.

The homogeneous systems that arise when you are finding eigenvectors will always row reduce to a matrix having at least one row of zeros, because the systems must have nontrivial solutions. A summary of the steps used to find the eigenvalues and corresponding eigenvectors of a matrix is below.

Finding Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.

1. Form the characteristic equation $|\lambda I - A| = 0$. It will be a polynomial equation of degree n in the variable λ .
2. Find the real roots of the characteristic equation. These are the eigenvalues of A .
3. For each eigenvalue λ_i , find the eigenvectors corresponding to λ_i by solving the homogeneous system $(\lambda_i I - A)\mathbf{x} = \mathbf{0}$. This can require row reducing an $n \times n$ matrix. The reduced row-echelon form must have at least one row of zeros.

Finding the eigenvalues of an $n \times n$ matrix can involve the factorization of an n th-degree polynomial. Once you have found an eigenvalue, you can find the corresponding eigenvectors by any appropriate method, such as Gauss-Jordan elimination.

EXAMPLE 5

Finding Eigenvalues and Eigenvectors

Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

What is the dimension of the eigenspace of each eigenvalue?

SOLUTION

The characteristic polynomial of A is

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} \\ &= (\lambda - 2)^3. \end{aligned}$$

So, the characteristic equation is $(\lambda - 2)^3 = 0$, and the only eigenvalue is $\lambda = 2$. To find the eigenvectors of $\lambda = 2$, solve the homogeneous linear system represented by $(2I - A)\mathbf{x} = \mathbf{0}$.

$$2I - A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This implies that $x_2 = 0$. Using the parameters $s = x_1$ and $t = x_3$, you can conclude that the eigenvectors of $\lambda = 2$ are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ } s \text{ and } t \text{ not both zero.}$$

$\lambda = 2$ has two linearly independent eigenvectors, so the dimension of its eigenspace is 2.



If an eigenvalue λ_i occurs as a *multiple root* (k times) of the characteristic polynomial, then λ_i has **multiplicity** k . This implies that $(\lambda - \lambda_i)^k$ is a factor of the characteristic polynomial and $(\lambda - \lambda_i)^{k+1}$ is not a factor of the characteristic polynomial. For instance, in Example 5, the eigenvalue $\lambda = 2$ has a multiplicity of 3.

Also note that in Example 5, the dimension of the eigenspace of $\lambda = 2$ is 2. In general, the multiplicity of an eigenvalue is greater than or equal to the dimension of its eigenspace. (In Exercise 63, you are asked to prove this.)

EXAMPLE 6

Finding Eigenvalues and Eigenvectors

Find the eigenvalues of

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

and find a basis for each of the corresponding eigenspaces.

SOLUTION

The characteristic polynomial of A is

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - 1 & 0 & 0 & 0 \\ 0 & \lambda - 1 & -5 & 10 \\ -1 & 0 & \lambda - 2 & 0 \\ -1 & 0 & 0 & \lambda - 3 \end{vmatrix} \\ &= (\lambda - 1)^2(\lambda - 2)(\lambda - 3). \end{aligned}$$

So, the characteristic equation is $(\lambda - 1)^2(\lambda - 2)(\lambda - 3) = 0$ and the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. (Note that $\lambda_1 = 1$ has a multiplicity of 2.)

You can find a basis for the eigenspace of $\lambda_1 = 1$ as shown below.

$$(1)I - A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Letting $s = x_2$ and $t = x_4$ produces

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0s - 2t \\ s + 0t \\ 0s + 2t \\ 0s + t \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

So, a basis for the eigenspace corresponding to $\lambda_1 = 1$ is

$$B_1 = \{(0, 1, 0, 0), (-2, 0, 2, 1)\}. \quad \text{Basis for } \lambda_1 = 1$$

For $\lambda_2 = 2$ and $\lambda_3 = 3$, use the same procedure to obtain the eigenspace bases

$$B_2 = \{(0, 5, 1, 0)\} \quad \text{Basis for } \lambda_2 = 2$$

$$B_3 = \{(0, -5, 0, 1)\}. \quad \text{Basis for } \lambda_3 = 3$$

TECHNOLOGY

Use a graphing utility or a software program to find the eigenvalues and eigenvectors in Example 6. When finding the eigenvectors, the technology you use may produce a matrix in which the columns are scalar multiples of the eigenvectors you would obtain by hand calculations. The **Technology Guide** at CengageBrain.com can help you use technology to find eigenvalues and eigenvectors.

Finding eigenvalues and eigenvectors of matrices of order $n \geq 4$ can be tedious. Moreover, using the procedure shown in Example 6 on a computer can introduce roundoff errors. Consequently, it can be more efficient to use numerical methods of approximating eigenvalues. One of these numerical methods appears in Section 10.3. Other methods appear in texts on advanced linear algebra and numerical analysis.

The next theorem states that the eigenvalues of an $n \times n$ triangular matrix are simply the entries on the main diagonal. The proof of this theorem follows from the fact that the determinant of a triangular matrix is the product of its main diagonal entries.

THEOREM 7.3 Eigenvalues of Triangular Matrices

If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

EXAMPLE 7

Finding Eigenvalues of Triangular and Diagonal Matrices

Find the eigenvalues of each matrix.

$$\begin{aligned} \text{a. } A &= \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix} \\ \text{b. } A &= \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \end{aligned}$$

SOLUTION

a. Without using Theorem 7.3,

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 3).$$

So, the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_3 = -3$, which are the main diagonal entries of A .

b. In this case, use Theorem 7.3 to conclude that the eigenvalues are the main diagonal entries $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 0$, $\lambda_4 = -4$, and $\lambda_5 = 3$. 



LINEAR ALGEBRA APPLIED

Eigenvalues and eigenvectors are useful for modeling real-life phenomena. For example, consider an experiment to determine the diffusion of a fluid from one flask to another through a permeable membrane and then out of the second flask. If researchers determine that the flow rate between flasks is twice the volume of fluid in the first flask and the flow rate out of the second flask is three times the volume of fluid in the second flask, then the system of linear differential equations below, where y_i represents the volume of fluid in flask i , models this situation.

$$\begin{aligned} y_1' &= -2y_1 \\ y_2' &= 2y_1 - 3y_2 \end{aligned}$$

In Section 7.4, you will use eigenvalues and eigenvectors to solve such systems of linear differential equations. For now, verify that the solution of this system is

$$\begin{aligned} y_1 &= C_1 e^{-2t} \\ y_2 &= 2C_1 e^{-2t} + C_2 e^{-3t}. \end{aligned}$$

Shi Yali/Shutterstock.com

EIGENVALUES AND EIGENVECTORS OF LINEAR TRANSFORMATIONS

This section began with definitions of eigenvalues and eigenvectors in terms of matrices. Eigenvalues and eigenvectors can also be defined in terms of linear transformations. A number λ is an **eigenvalue** of a linear transformation $T: V \rightarrow V$ when there is a nonzero vector $\mathbf{x}$ such that $T(\mathbf{x}) = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is an **eigenvector** of T corresponding to λ , and the set of all eigenvectors of λ (with the zero vector) is the **eigenspace** of λ .

Consider $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, whose matrix relative to the standard basis is

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \quad \begin{array}{l} \text{Standard basis:} \\ B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \end{array}$$

In Example 5 in Section 6.4, you found that the matrix of T relative to the basis $B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$ is the diagonal matrix

$$A' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \quad \begin{array}{l} \text{Nonstandard basis:} \\ B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\} \end{array}$$

For a linear transformation T , can you find a basis B' whose corresponding matrix is diagonal? The next example illustrates the answer.

EXAMPLE 8

Finding Eigenvalues and Eigenspaces

Find the eigenvalues and a basis for each corresponding eigenspace of

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

SOLUTION

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix} \\ &= [(\lambda - 1)^2 - 9](\lambda + 2) \\ &= (\lambda - 4)(\lambda + 2)^2 \end{aligned}$$

so the eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = -2$. Bases for the eigenspaces are $B_1 = \{(1, 1, 0)\}$ and $B_2 = \{(1, -1, 0), (0, 0, 1)\}$, respectively (verify these).

Example 8 illustrates two results. If $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the linear transformation whose standard matrix is A , and B' is a basis for $\mathbb{R}^3$ made up of the three linearly independent eigenvectors corresponding to the eigenvalues of A , then the matrix A' for T relative to the basis B' is diagonal. Also, the main diagonal entries of the matrix A' are the eigenvalues of A .

$$A' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \begin{array}{l} \text{Nonstandard basis:} \\ B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\} \end{array}$$



Eigenvalues of A



Eigenvectors of A

The next section discusses these results in more detail.

7.1 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Verifying Eigenvalues and Eigenvectors In Exercises 1–6, verify that λ_i is an eigenvalue of A and that $\mathbf{x}_i$ is a corresponding eigenvector.

1. $A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$, $\lambda_1 = 2$, $\mathbf{x}_1 = (1, 0)$
 $\lambda_2 = -2$, $\mathbf{x}_2 = (0, 1)$
2. $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$, $\lambda_1 = -1$, $\mathbf{x}_1 = (1, 1)$
 $\lambda_2 = 2$, $\mathbf{x}_2 = (5, 2)$
3. $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, $\lambda_1 = 2$, $\mathbf{x}_1 = (1, 0, 0)$
 $\lambda_2 = -1$, $\mathbf{x}_2 = (1, -1, 0)$
 $\lambda_3 = 3$, $\mathbf{x}_3 = (5, 1, 2)$
4. $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$, $\lambda_1 = 5$, $\mathbf{x}_1 = (1, 2, -1)$
 $\lambda_2 = -3$, $\mathbf{x}_2 = (-2, 1, 0)$
 $\lambda_3 = -3$, $\mathbf{x}_3 = (3, 0, 1)$
5. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $\lambda_1 = 1$, $\mathbf{x}_1 = (1, 1, 1)$
6. $A = \begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, $\lambda_1 = 4$, $\mathbf{x}_1 = (1, 0, 0)$
 $\lambda_2 = 2$, $\mathbf{x}_2 = (1, 2, 0)$
 $\lambda_3 = 3$, $\mathbf{x}_3 = (-2, 1, 1)$
7. Use A , λ_i , and $\mathbf{x}_i$ from Exercise 1 to show that
 - (a) $A(c\mathbf{x}_1) = 2(c\mathbf{x}_1)$ for any real number c .
 - (b) $A(c\mathbf{x}_2) = -2(c\mathbf{x}_2)$ for any real number c .
8. Use A , λ_i , and $\mathbf{x}_i$ from Exercise 4 to show that
 - (a) $A(c\mathbf{x}_1) = 5(c\mathbf{x}_1)$ for any real number c .
 - (b) $A(c\mathbf{x}_2) = -3(c\mathbf{x}_2)$ for any real number c .
 - (c) $A(c\mathbf{x}_3) = -3(c\mathbf{x}_3)$ for any real number c .

Determining Eigenvectors In Exercises 9–12, determine whether $\mathbf{x}$ is an eigenvector of A .

9. $A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$
 - (a) $\mathbf{x} = (1, 2)$
 - (b) $\mathbf{x} = (2, 1)$
 - (c) $\mathbf{x} = (1, -2)$
 - (d) $\mathbf{x} = (-1, 0)$
10. $A = \begin{bmatrix} -3 & 10 \\ 5 & 2 \end{bmatrix}$
 - (a) $\mathbf{x} = (4, 4)$
 - (b) $\mathbf{x} = (-8, 4)$
 - (c) $\mathbf{x} = (-4, 8)$
 - (d) $\mathbf{x} = (5, -3)$
11. $A = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix}$
 - (a) $\mathbf{x} = (2, -4, 6)$
 - (b) $\mathbf{x} = (2, 0, 6)$
 - (c) $\mathbf{x} = (2, 2, 0)$
 - (d) $\mathbf{x} = (-1, 0, 1)$
12. $A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & -2 & 4 \\ 1 & -2 & 9 \end{bmatrix}$
 - (a) $\mathbf{x} = (1, 1, 0)$
 - (b) $\mathbf{x} = (-5, 2, 1)$
 - (c) $\mathbf{x} = (0, 0, 0)$
 - (d) $\mathbf{x} = (2\sqrt{6} - 3, -2\sqrt{6} + 6, 3)$

Finding Eigenspaces in $\mathbb{R}^2$ Geometrically In Exercises 13 and 14, use the method shown in Example 3 to find the eigenvalue(s) and corresponding eigenspace(s) of A .

$$13. A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad 14. A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Characteristic Equation, Eigenvalues, and Eigenvectors In Exercises 15–28, find (a) the characteristic equation and (b) the eigenvalues (and corresponding eigenvectors) of the matrix.

15. $\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$
16. $\begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}$
17. $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
18. $\begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}$
19. $\begin{bmatrix} 1 & -\frac{3}{2} \\ \frac{1}{2} & -1 \end{bmatrix}$
20. $\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 \end{bmatrix}$
21. $\begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$
22. $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$
23. $\begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$
24. $\begin{bmatrix} 3 & 2 & -3 \\ -3 & -4 & 9 \\ -1 & -2 & 5 \end{bmatrix}$
25. $\begin{bmatrix} 0 & -3 & 5 \\ -4 & 4 & -10 \\ 0 & 0 & 4 \end{bmatrix}$
26. $\begin{bmatrix} 1 & -\frac{3}{2} & \frac{5}{2} \\ -2 & \frac{13}{2} & -10 \\ \frac{3}{2} & -\frac{9}{2} & 8 \end{bmatrix}$
27. $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 4 & 0 \end{bmatrix}$
28. $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$



Finding Eigenvalues In Exercises 29–40, use a software program or a graphing utility to find the eigenvalues of the matrix.

29. $\begin{bmatrix} -4 & 5 \\ -2 & 3 \end{bmatrix}$
30. $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$
31. $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$
32. $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$
33. $\begin{bmatrix} 2 & 4 & 2 \\ 1 & 0 & 1 \\ 1 & -4 & 5 \end{bmatrix}$
34. $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$
35. $\begin{bmatrix} 3 & -\frac{1}{2} & 5 \\ -\frac{1}{3} & -\frac{1}{6} & -\frac{1}{4} \\ 0 & 0 & 4 \end{bmatrix}$
36. $\begin{bmatrix} \frac{1}{2} & 0 & 5 \\ -2 & \frac{1}{5} & \frac{1}{4} \\ 1 & 0 & 3 \end{bmatrix}$

$$37. \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 4 & 6 \\ 3 & 3 & 6 & 9 \\ 4 & 4 & 8 & 12 \end{bmatrix} \quad 38. \begin{bmatrix} 1 & 1 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$39. \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ -2 & 0 & 2 & -2 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$

$$40. \begin{bmatrix} 1 & -3 & 3 & 3 \\ -1 & 4 & -3 & -3 \\ -2 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvalues of Triangular and Diagonal Matrices In Exercises 41–44, find the eigenvalues of the triangular or diagonal matrix.

$$41. \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$42. \begin{bmatrix} -5 & 0 & 0 \\ 3 & 7 & 0 \\ 4 & -2 & 3 \end{bmatrix}$$

$$43. \begin{bmatrix} -6 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$44. \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{5}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{4} \end{bmatrix}$$

Eigenvalues and Eigenvectors of Linear Transformations In Exercises 45–48, consider the linear transformation $T: R^n \rightarrow R^n$ whose matrix A relative to the standard basis is given. Find (a) the eigenvalues of A , (b) a basis for each of the corresponding eigenspaces, and (c) the matrix A' for T relative to the basis B' , where B' is made up of the basis vectors found in part (b).

$$45. \begin{bmatrix} 2 & -2 \\ 1 & 5 \end{bmatrix}$$

$$46. \begin{bmatrix} -8 & 16 \\ 1 & -2 \end{bmatrix}$$

$$47. \begin{bmatrix} 0 & 2 & -1 \\ -1 & 3 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$48. \begin{bmatrix} 3 & 1 & 4 \\ 2 & 4 & 0 \\ 5 & 5 & 6 \end{bmatrix}$$

Cayley-Hamilton Theorem In Exercises 49–52, demonstrate the Cayley-Hamilton Theorem for the matrix A . The Cayley-Hamilton Theorem states that a matrix satisfies its characteristic equation. For example, the characteristic equation of

$$A = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}$$

is $\lambda^2 - 6\lambda + 11 = 0$, and by the theorem you have $A^2 - 6A + 11I_2 = O$.

$$49. A = \begin{bmatrix} 5 & 0 \\ -7 & 3 \end{bmatrix}$$

$$50. A = \begin{bmatrix} 6 & -1 \\ 1 & 5 \end{bmatrix}$$

$$51. A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$52. A = \begin{bmatrix} -3 & 1 & 0 \\ -1 & 3 & 2 \\ 0 & 4 & 3 \end{bmatrix}$$

53. Perform each computational check on the eigenvalues found in Exercises 15–27 odd.

(a) The sum of the n eigenvalues equals the trace of the matrix. (Recall that the **trace** of a matrix is the sum of the main diagonal entries of the matrix.)

(b) The product of the n eigenvalues equals $|A|$.

(When λ is an eigenvalue of multiplicity k , remember to use it k times in the sum or product of these checks.)

54. Perform each computational check on the eigenvalues found in Exercises 16–28 even.

(a) The sum of the n eigenvalues equals the trace of the matrix. (Recall that the **trace** of a matrix is the sum of the main diagonal entries of the matrix.)

(b) The product of the n eigenvalues equals $|A|$.

(When λ is an eigenvalue of multiplicity k , remember to use it k times in the sum or product of these checks.)

55. Show that if A is an $n \times n$ matrix whose i th row is identical to the i th row of I , then 1 is an eigenvalue of A .

56. **Proof** Prove that $\lambda = 0$ is an eigenvalue of A if and only if A is singular.

57. **Proof** For an invertible matrix A , prove that A and A^{-1} have the same eigenvectors. How are the eigenvalues of A related to the eigenvalues of A^{-1} ?

58. **Proof** Prove that A and A^T have the same eigenvalues. Are the eigenspaces the same?

59. **Proof** Prove that the constant term of the characteristic polynomial is $\pm|A|$.

60. Define $T: R^2 \rightarrow R^2$ by

$$T(\mathbf{v}) = \text{proj}_{\mathbf{u}} \mathbf{v}$$

where $\mathbf{u}$ is a fixed vector in R^2 . Show that the eigenvalues of A (the standard matrix of T) are 0 and 1.

61. **Guided Proof** Prove that a triangular matrix is nonsingular if and only if its eigenvalues are real and nonzero.

Getting Started: This is an “if and only if” statement, so you must prove that the statement is true in both directions. Review Theorems 3.2 and 3.7.

(i) To prove the statement in one direction, assume that the triangular matrix A is nonsingular. Use your knowledge of nonsingular and triangular matrices and determinants to conclude that the entries on the main diagonal of A are nonzero.

(ii) A is triangular, so use Theorem 7.3 and part (i) to conclude that the eigenvalues are real and nonzero.

(iii) To prove the statement in the other direction, assume that the eigenvalues of the triangular matrix A are real and nonzero. Repeat parts (i) and (ii) in reverse order to prove that A is nonsingular.

- 62. Guided Proof** Prove that if $A^2 = O$, then 0 is the only eigenvalue of A .

Getting Started: You need to show that if there exists a nonzero vector $\mathbf{x}$ and a real number λ such that $A\mathbf{x} = \lambda\mathbf{x}$, then if $A^2 = O$, λ must be zero.

- (i) $A^2 = A \cdot A$, so you can write $A^2\mathbf{x}$ as $A(A\mathbf{x})$.
- (ii) Use the fact that $A\mathbf{x} = \lambda\mathbf{x}$ and the properties of matrix multiplication to show that $A^2\mathbf{x} = \lambda^2\mathbf{x}$.
- (iii) A^2 is a zero matrix, so you can conclude that λ must be zero.

- 63. Proof** Prove that the multiplicity of an eigenvalue is greater than or equal to the dimension of its eigenspace.

- 64. CAPSTONE** An $n \times n$ matrix A has the characteristic equation

$$|\lambda I - A| = (\lambda + 2)(\lambda - 1)(\lambda - 3)^2 = 0.$$

- (a) What are the eigenvalues of A ?
- (b) What is the order of A ? Explain.
- (c) Is $\lambda I - A$ singular? Explain.
- (d) Is A singular? Explain. (*Hint*: Use the result of Exercise 56.)

- 65.** When the eigenvalues of

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

are $\lambda_1 = 0$ and $\lambda_2 = 1$, what are the possible values of a and d ?

- 66.** Show that

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

has no real eigenvalues.

True or False? In Exercises 67 and 68, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

- 67.** (a) The scalar λ is an eigenvalue of an $n \times n$ matrix A when there exists a vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$.
(b) To find the eigenvalue(s) of an $n \times n$ matrix A , you can solve the characteristic equation $\det(\lambda I - A) = 0$.
- 68.** (a) Geometrically, if λ is an eigenvalue of a matrix A and $\mathbf{x}$ is an eigenvector of A corresponding to λ , then multiplying $\mathbf{x}$ by A produces a vector $\lambda\mathbf{x}$ parallel to $\mathbf{x}$.
(b) If A is an $n \times n$ matrix with an eigenvalue λ , then the set of all eigenvectors of λ is a subspace of R^n .

Finding the Dimension of an Eigenspace In Exercises 69–72, find the dimension of the eigenspace corresponding to the eigenvalue $\lambda = 3$.

69. $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ **70.** $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

71. $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ **72.** $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

- 73. Calculus** Let $T: C'[0, 1] \rightarrow C[0, 1]$ be the linear transformation $T(f) = f'$. Show that $\lambda = 1$ is an eigenvalue of T with corresponding eigenvector $f(x) = e^x$.

- 74. Calculus** For the linear transformation in Exercise 73, find the eigenvalue corresponding to the eigenvector $f(x) = e^{-2x}$.

- 75.** Define $T: P_2 \rightarrow P_2$ by

$$T(a_0 + a_1x + a_2x^2) = (-3a_1 + 5a_2) + (-4a_0 + 4a_1 - 10a_2)x + 4a_2x^2.$$

Find the eigenvalues and the eigenvectors of T relative to the standard basis $\{1, x, x^2\}$.

- 76.** Define $T: P_2 \rightarrow P_2$ by

$$T(a_0 + a_1x + a_2x^2) = (2a_0 + a_1 - a_2) + (-a_1 + 2a_2)x - a_2x^2.$$

Find the eigenvalues and eigenvectors of T relative to the standard basis $\{1, x, x^2\}$.

- 77.** Define $T: M_{2,2} \rightarrow M_{2,2}$ by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a - c + d & b + d \\ -2a + 2c - 2d & 2b + 2d \end{bmatrix}$$

Find the eigenvalues and eigenvectors of T relative to the standard basis

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

- 78.** Find all values of the angle θ for which the matrix

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

has real eigenvalues. Interpret your answer geometrically.

- 79.** What are the possible eigenvalues of an idempotent matrix? (Recall that a square matrix A is **idempotent** when $A^2 = A$.)
- 80.** What are the possible eigenvalues of a nilpotent matrix? (Recall that a square matrix A is **nilpotent** when there exists a positive integer k such that $A^k = O$.)
- 81. Proof** Let A be an $n \times n$ matrix such that the sum of the entries in each row is a fixed constant r . Prove that r is an eigenvalue of A . Illustrate this result with an example.

7.2 Diagonalization

-  Find the eigenvalues of similar matrices, determine whether a matrix A is diagonalizable, and find a matrix P such that $P^{-1}AP$ is diagonal.
-  Find, for a linear transformation $T: V \rightarrow V$, a basis B for V such that the matrix for T relative to B is diagonal.

THE DIAGONALIZATION PROBLEM

The preceding section discussed the eigenvalue problem. In this section, you will look at another classic problem in linear algebra called the **diagonalization problem**. Expressed in terms of matrices*, the problem is “for a square matrix A , does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?”

Recall from Section 6.4 that two square matrices A and B are similar when there exists an invertible matrix P such that $B = P^{-1}AP$.

Matrices that are similar to diagonal matrices are called **diagonalizable**.

Definition of a Diagonalizable Matrix

An $n \times n$ matrix A is **diagonalizable** when A is similar to a diagonal matrix. That is, A is diagonalizable when there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

With this definition, the diagonalization problem can be stated as “which square matrices are diagonalizable?” Clearly, every diagonal matrix D is diagonalizable, because $D = I^{-1}DI$, where I is the identity matrix. Example 1 shows another example of a diagonalizable matrix.

EXAMPLE 1

A Diagonalizable Matrix

The matrix from Example 5 in Section 6.4

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

is diagonalizable because

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has the property that

$$P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

As suggested in Example 8 in the preceding section, the eigenvalue problem is related closely to the diagonalization problem. The next two theorems shed more light on this relationship. The first theorem tells you that similar matrices have the same eigenvalues.

*At the end of this section, the diagonalization problem will be expressed in terms of linear transformations.

THEOREM 7.4 Similar Matrices Have the Same Eigenvalues

If A and B are similar $n \times n$ matrices, then they have the same eigenvalues.

PROOF

A and B are similar, so there exists an invertible matrix P such that $B = P^{-1}AP$. By the properties of determinants, it follows that

$$\begin{aligned} |\lambda I - B| &= |\lambda I - P^{-1}AP| \\ &= |P^{-1}\lambda IP - P^{-1}AP| \\ &= |P^{-1}(\lambda I - A)P| \\ &= |P^{-1}| |\lambda I - A| |P| \\ &= |P^{-1}P| |\lambda I - A| \\ &= |\lambda I - A|. \end{aligned}$$

This means that A and B have the same characteristic polynomial. So, they must have the same eigenvalues. 

REMARK

Check that A and D are similar by showing that they satisfy the matrix equation $D = P^{-1}AP$, where

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

In fact, the columns of P are eigenvectors of A corresponding to the eigenvalues 1, 2, and 3. (Verify this.)

EXAMPLE 2**Finding Eigenvalues of Similar Matrices**

The matrices A and D are similar.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Use Theorem 7.4 to find the eigenvalues of A .

SOLUTION

D is a diagonal matrix, so its eigenvalues are the entries on its main diagonal—that is, $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. Matrices A and D are similar, so you know from Theorem 7.4 that A has the same eigenvalues. Check this by showing that the characteristic polynomial of A is $|\lambda I - A| = (\lambda - 1)(\lambda - 2)(\lambda - 3)$. 

The two diagonalizable matrices in Examples 1 and 2 provide a clue to the diagonalization problem. Each of these matrices has a set of three linearly independent eigenvectors. (See Example 3.) This is characteristic of diagonalizable matrices, as stated in Theorem 7.5.

THEOREM 7.5 Condition for Diagonalization

An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

PROOF

First, assume A is diagonalizable. Then there exists an invertible matrix P such that $P^{-1}AP = D$ is diagonal. Letting the column vectors of P be $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$, and the main diagonal entries of D be $\lambda_1, \lambda_2, \dots, \lambda_n$, produces

$$PD = [\mathbf{p}_1 \ \mathbf{p}_2 \ \dots \ \mathbf{p}_n] \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} = [\lambda_1 \mathbf{p}_1 \ \lambda_2 \mathbf{p}_2 \ \dots \ \lambda_n \mathbf{p}_n].$$

$P^{-1}AP = D$, so $AP = PD$, which implies

$$[A\mathbf{p}_1 \ A\mathbf{p}_2 \ \dots \ A\mathbf{p}_n] = [\lambda_1\mathbf{p}_1 \ \lambda_2\mathbf{p}_2 \ \dots \ \lambda_n\mathbf{p}_n].$$

In other words, $A\mathbf{p}_i = \lambda_i\mathbf{p}_i$ for each column vector $\mathbf{p}_i$. This means that the column vectors $\mathbf{p}_i$ of P are eigenvectors of A . Moreover, P is invertible, so its column vectors are linearly independent. So, A has n linearly independent eigenvectors.

Conversely, assume A has n linearly independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Let P be the matrix whose columns are these n eigenvectors. That is, $P = [\mathbf{p}_1 \ \mathbf{p}_2 \ \dots \ \mathbf{p}_n]$. Each $\mathbf{p}_i$ is an eigenvector of A , so you have $A\mathbf{p}_i = \lambda_i\mathbf{p}_i$ and

$$AP = A[\mathbf{p}_1 \ \mathbf{p}_2 \ \dots \ \mathbf{p}_n] = [\lambda_1\mathbf{p}_1 \ \lambda_2\mathbf{p}_2 \ \dots \ \lambda_n\mathbf{p}_n].$$

The right-hand matrix in this equation can be written as the matrix product below.

$$[\lambda_1\mathbf{p}_1 \ \lambda_2\mathbf{p}_2 \ \dots \ \lambda_n\mathbf{p}_n] = [\mathbf{p}_1 \ \mathbf{p}_2 \ \dots \ \mathbf{p}_n] \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} = PD$$

Finally, the vectors $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$ are linearly independent, so P is invertible and you can write the equation $AP = PD$ as $P^{-1}AP = D$, which means that A is diagonalizable. 

A key result of this proof is the fact that for diagonalizable matrices, *the columns of P consist of n linearly independent eigenvectors*. Example 3 verifies this important property for the matrices in Examples 1 and 2.

EXAMPLE 3

Diagonalizable Matrices

- a. The matrix A in Example 1 has the eigenvalues and corresponding eigenvectors below.

$$\lambda_1 = 4, \mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \quad \lambda_2 = -2, \mathbf{p}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}; \quad \lambda_3 = -2, \mathbf{p}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The matrix P whose columns correspond to these eigenvectors is

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Moreover, P is row-equivalent to the identity matrix, so the eigenvectors $\mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent.

- b. The matrix A in Example 2 has the eigenvalues and corresponding eigenvectors below.

$$\lambda_1 = 1, \mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \quad \lambda_2 = 2, \mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}; \quad \lambda_3 = 3, \mathbf{p}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

The matrix P whose columns correspond to these eigenvectors is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Again, P is row-equivalent to the identity matrix, so the eigenvectors $\mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent. 

The second part of the proof of Theorem 7.5 and Example 3 suggest the steps for diagonalizing a matrix listed below.

Steps for Diagonalizing a Square Matrix

Let A be an $n \times n$ matrix.

1. Find n linearly independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$ for A (if possible) with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. If n linearly independent eigenvectors do not exist, then A is not diagonalizable.
2. Let P be the $n \times n$ matrix whose columns consist of these eigenvectors. That is, $P = [\mathbf{p}_1 \ \mathbf{p}_2 \ \dots \ \mathbf{p}_n]$.
3. The diagonal matrix $D = P^{-1}AP$ will have the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ on its main diagonal. Note that the order of the eigenvectors used to form P will determine the order in which the eigenvalues appear on the main diagonal of D .

EXAMPLE 4 Diagonalizing a Matrix

Show that the matrix A is diagonalizable.

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Then find a matrix P such that $P^{-1}AP$ is diagonal.

SOLUTION

The characteristic polynomial of A is $|\lambda I - A| = (\lambda - 2)(\lambda + 2)(\lambda - 3)$. (Verify this.) So, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = -2$, and $\lambda_3 = 3$. From these eigenvalues, you obtain the reduced row-echelon forms and corresponding eigenvectors below.

	→		Eigenvector
$2I - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix}$	→	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
$-2I - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix}$	→	$\begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$
$3I - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix}$	→	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Form the matrix P whose columns are the eigenvectors just obtained.

$$P = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

This matrix is nonsingular (check this), which implies that the eigenvectors are linearly independent and A is diagonalizable. So, it follows that

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$



EXAMPLE 5**Diagonalizing a Matrix**

Show that the matrix A is diagonalizable.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

Then find a matrix P such that $P^{-1}AP$ is diagonal.

SOLUTION

In Example 6 in Section 7.1, you found that the three eigenvalues of A are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$, and that they have the eigenvectors listed below.

$$\lambda_1: \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \quad \lambda_2: \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \quad \lambda_3: \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

The matrix whose columns consist of these eigenvectors is

$$P = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 1 & 0 & 5 & -5 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

P is invertible (check this), so its column vectors form a linearly independent set. This means that A is diagonalizable, and

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

**EXAMPLE 6****A Matrix That Is Not Diagonalizable**

Show that the matrix A is not diagonalizable.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

SOLUTION

A is triangular, so the eigenvalues are the entries on the main diagonal. The only eigenvalue is $\lambda = 1$. The matrix $(I - A)$ has the reduced row-echelon form below.

$$I - A = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This implies that $x_2 = 0$, and letting $x_1 = t$, you can write every eigenvector of A in the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

So, A does not have two linearly independent eigenvectors, and you can conclude that A is not diagonalizable.



REMARK

The condition in Theorem 7.6 is sufficient but not necessary for diagonalization, as demonstrated in Example 5. In other words, a diagonalizable matrix need not have distinct eigenvalues.

For a square matrix A of order n to be diagonalizable, the sum of the dimensions of the eigenspaces must be equal to n . This can happen when A has n distinct eigenvalues. So, you have the next theorem.

THEOREM 7.6 Sufficient Condition for Diagonalization

If an $n \times n$ matrix A has n *distinct* eigenvalues, then the corresponding eigenvectors are linearly independent and A is diagonalizable.

PROOF

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be n distinct eigenvalues of A with corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$. To begin, assume the set of eigenvectors is linearly dependent. Moreover, consider the eigenvectors to be ordered so that the first m eigenvectors are linearly independent, but the first $m + 1$ are linearly dependent, where $m < n$. Then $\mathbf{x}_{m+1}$ can be written as a linear combination of the first m eigenvectors:

$$\mathbf{x}_{m+1} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_m\mathbf{x}_m \quad \text{Equation 1}$$

where the c_i 's are not all zero. Multiplication of both sides of Equation 1 by A yields

$$A\mathbf{x}_{m+1} = Ac_1\mathbf{x}_1 + Ac_2\mathbf{x}_2 + \cdots + Ac_m\mathbf{x}_m.$$

Now $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$, $i = 1, 2, \dots, m + 1$, so you have

$$\lambda_{m+1}\mathbf{x}_{m+1} = c_1\lambda_1\mathbf{x}_1 + c_2\lambda_2\mathbf{x}_2 + \cdots + c_m\lambda_m\mathbf{x}_m. \quad \text{Equation 2}$$

Multiplication of Equation 1 by λ_{m+1} yields

$$\lambda_{m+1}\mathbf{x}_{m+1} = c_1\lambda_{m+1}\mathbf{x}_1 + c_2\lambda_{m+1}\mathbf{x}_2 + \cdots + c_m\lambda_{m+1}\mathbf{x}_m. \quad \text{Equation 3}$$

Subtracting Equation 2 from Equation 3 produces

$$c_1(\lambda_{m+1} - \lambda_1)\mathbf{x}_1 + c_2(\lambda_{m+1} - \lambda_2)\mathbf{x}_2 + \cdots + c_m(\lambda_{m+1} - \lambda_m)\mathbf{x}_m = \mathbf{0}$$

and, using the fact that the first m eigenvectors are linearly independent, all coefficients of this equation must be zero. That is,

$$c_1(\lambda_{m+1} - \lambda_1) = c_2(\lambda_{m+1} - \lambda_2) = \cdots = c_m(\lambda_{m+1} - \lambda_m) = 0.$$

All the eigenvalues are distinct, so it follows that $c_i = 0$, $i = 1, 2, \dots, m$. But this result contradicts our assumption that $\mathbf{x}_{m+1}$ can be written as a linear combination of the first m eigenvectors. So, the set of eigenvectors is linearly independent, and from Theorem 7.5, you can conclude that A is diagonalizable. 

EXAMPLE 7**Determining Whether a Matrix Is Diagonalizable**

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Determine whether the matrix A is diagonalizable.

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

SOLUTION

A is a triangular matrix, so its eigenvalues are the main diagonal entries $\lambda_1 = 1$, $\lambda_2 = 0$, and $\lambda_3 = -3$. Moreover, these three values are distinct, so you can conclude from Theorem 7.6 that A is diagonalizable. 

DIAGONALIZATION AND LINEAR TRANSFORMATIONS

So far in this section, the diagonalization problem has been in terms of matrices. In terms of linear transformations, the diagonalization problem can be stated as: For a linear transformation

$$T: V \rightarrow V$$

does there exist a basis B for V such that the matrix for T relative to B is diagonal? The answer is “yes” when the standard matrix for T is diagonalizable.

EXAMPLE 8 Finding a Basis

Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation represented by

$$T(x_1, x_2, x_3) = (x_1 - x_2 - x_3, x_1 + 3x_2 + x_3, -3x_1 + x_2 - x_3).$$

If possible, find a basis B for $\mathbb{R}^3$ such that the matrix for T relative to B is diagonal.

SOLUTION

The standard matrix for T is

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}.$$

From Example 4, you know that A is diagonalizable. So, the three linearly independent eigenvectors found in Example 4 can be used to form the basis B . That is,

$$B = \{(-1, 0, 1), (1, -1, 4), (-1, 1, 1)\}.$$

The matrix for T relative to this basis is

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$



LINEAR ALGEBRA APPLIED

Genetics is the science of heredity. A mixture of chemistry and biology, genetics attempts to explain hereditary evolution and gene movement between generations based on the deoxyribonucleic acid (DNA) of a species. Research in the area of genetics called *population genetics*, which focuses on genetic structures of specific populations, is especially popular today. Such research has led to a better understanding of the types of genetic inheritance. For example, in humans, one type of genetic inheritance is called *X-linked inheritance* (or *sex-linked inheritance*), which refers to recessive genes on the X chromosome. Males have one X and one Y chromosome, and females have two X chromosomes. If a male has a defective gene on the X chromosome, then its corresponding trait will be expressed because there is not a normal gene on the Y chromosome to suppress its activity. With females, the trait will not be expressed unless it is present on both X chromosomes, which is rare. This is why inherited diseases or conditions are usually found in males, hence the term *sex-linked inheritance*. Some of these include hemophilia A, Duchenne muscular dystrophy, red-green color blindness, and male pattern baldness. Matrix eigenvalues and diagonalization can be useful for coming up with mathematical models to describe X-linked inheritance in a population.

Sergey Nivens/Shutterstock.com

7.2 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Diagonalizable Matrices and Eigenvalues In Exercises 1–6, (a) verify that A is diagonalizable by finding $P^{-1}AP$, and (b) use the result of part (a) and Theorem 7.4 to find the eigenvalues of A .

1. $A = \begin{bmatrix} -11 & 36 \\ -3 & 10 \end{bmatrix}, P = \begin{bmatrix} -3 & -4 \\ -1 & -1 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}, P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}, P = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}, P = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}$

5. $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 3 & 0 \\ 4 & -2 & 5 \end{bmatrix}, P = \begin{bmatrix} 0 & 1 & -3 \\ 0 & 4 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

6. $A = \begin{bmatrix} 0.80 & 0.10 & 0.05 & 0.05 \\ 0.10 & 0.80 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.80 & 0.10 \\ 0.05 & 0.05 & 0.10 & 0.80 \end{bmatrix},$

$P = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}$

Diagonalizing a Matrix In Exercises 7–14, find (if possible) a nonsingular matrix P such that $P^{-1}AP$ is diagonal. Verify that $P^{-1}AP$ is a diagonal matrix with the eigenvalues on the main diagonal.

7. $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

(See Exercise 15, Section 7.1.)

8. $A = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 \end{bmatrix}$

(See Exercise 20, Section 7.1.)

9. $A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$

(See Exercise 21, Section 7.1.)

10. $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

(See Exercise 22, Section 7.1.)

11. $A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$

(See Exercise 23, Section 7.1.)

12. $A = \begin{bmatrix} 3 & 2 & -3 \\ -3 & -4 & 9 \\ -1 & -2 & 5 \end{bmatrix}$

(See Exercise 24, Section 7.1.)

13. $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

14. $A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 4 & 0 \\ 0 & 4 & 4 \end{bmatrix}$

Showing That a Matrix Is Not Diagonalizable In Exercises 15–22, show that the matrix is not diagonalizable.

15. $\begin{bmatrix} 0 & 0 \\ 5 & 0 \end{bmatrix}$

16. $\begin{bmatrix} 1 & \frac{1}{2} \\ -2 & -1 \end{bmatrix}$

17. $\begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

20. $\begin{bmatrix} 3 & 2 & -2 \\ 0 & -2 & 3 \\ 0 & 0 & -2 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ -2 & 0 & 2 & -2 \\ 0 & 2 & 0 & 2 \end{bmatrix}$

(See Exercise 39, Section 7.1.)

22. $\begin{bmatrix} 1 & -3 & 3 & 3 \\ -1 & 4 & -3 & -3 \\ -2 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

(See Exercise 40, Section 7.1.)

Determining a Sufficient Condition for Diagonalization

In Exercises 23–26, find the eigenvalues of the matrix and determine whether there is a sufficient number of eigenvalues to guarantee that the matrix is diagonalizable by Theorem 7.6.

23. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

24. $\begin{bmatrix} 2 & 0 \\ 5 & 2 \end{bmatrix}$

25. $\begin{bmatrix} -3 & -2 & 3 \\ 3 & 4 & -9 \\ 1 & 2 & -5 \end{bmatrix}$

26. $\begin{bmatrix} 4 & 3 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$

Finding a Basis In Exercises 27–30, find a basis B for the domain of T such that the matrix for T relative to B is diagonal.

27. $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2: T(x, y) = (x + y, x + y)$

28. $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3:$

$T(x, y, z) = (-2x + 2y - 3z, 2x + y - 6z, -x - 2y)$

29. $T: P_1 \rightarrow P_1: T(a + bx) = a + (a + 2b)x$

30. $T: P_2 \rightarrow P_2:$

$T(c + bx + ax^2) = (3c + a) + (2b + 3a)x + ax^2$

31. **Proof** Let A be a diagonalizable $n \times n$ matrix and let P be an invertible $n \times n$ matrix such that $B = P^{-1}AP$ is the diagonal form of A . Prove that $A^k = PB^kP^{-1}$, where k is a positive integer.

32. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be n distinct eigenvalues of an $n \times n$ matrix A . Use the result of Exercise 31 to find the eigenvalues of A^k .

Finding a Power of a Matrix In Exercises 33–36, use the result of Exercise 31 to find the power of A shown.

33. $A = \begin{bmatrix} 10 & 18 \\ -6 & -11 \end{bmatrix}, A^6$ 34. $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}, A^7$

35. $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & -2 \\ 3 & 0 & -3 \end{bmatrix}, A^5$

36. $A = \begin{bmatrix} 2 & 3 & -2 \\ -2 & -5 & 0 \\ -2 & -1 & 4 \end{bmatrix}, A^8$

True or False? In Exercises 37 and 38, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

37. (a) If A and B are similar $n \times n$ matrices, then they always have the same characteristic polynomial equation.
 (b) The fact that an $n \times n$ matrix A has n distinct eigenvalues does not guarantee that A is diagonalizable.
38. (a) If A is a diagonalizable matrix, then it has n linearly independent eigenvectors.
 (b) If an $n \times n$ matrix A is diagonalizable, then it must have n distinct eigenvalues.
39. Are the two matrices similar? If so, find a matrix P such that $B = P^{-1}AP$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

40. **Calculus** For a real number x , you can define e^x by the series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

In a similar way, for a square matrix X , you can define e^X by the series

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \cdots$$

Evaluate e^X , where X is the square matrix shown.

(a) $X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $X = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$
 (c) $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (d) $X = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

41. **Writing** Can a matrix be similar to two different diagonal matrices? Explain.

42. **Proof** Prove that if matrix A is diagonalizable, then A^T is diagonalizable.

43. **Proof** Prove that if matrix A is diagonalizable with n real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then $|A| = \lambda_1 \lambda_2 \cdots \lambda_n$.

44. **Proof** Prove that the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is diagonalizable when $-4bc < (a - d)^2$ and is not diagonalizable when $-4bc > (a - d)^2$.

45. **Guided Proof** Prove that if the eigenvalues of a diagonalizable matrix A are all ± 1 , then the matrix is equal to its inverse.

Getting Started: To show that the matrix is equal to its inverse, use the fact that there exists an invertible matrix P such that $D = P^{-1}AP$, where D is a diagonal matrix with ± 1 along its main diagonal.

- (i) Let $D = P^{-1}AP$, where D is a diagonal matrix with ± 1 along its main diagonal.
 (ii) Find A in terms of P , P^{-1} , and D .
 (iii) Use the properties of the inverse of a product of matrices and the fact that D is diagonal to expand to find A^{-1} .
 (iv) Conclude that $A^{-1} = A$.

46. **Guided Proof** Prove that nonzero nilpotent matrices are not diagonalizable.

Getting Started: From Exercise 80 in Section 7.1, you know that 0 is the only eigenvalue of the nilpotent matrix A . Show that it is impossible for A to be diagonalizable.

- (i) Assume A is diagonalizable, so there exists an invertible matrix P such that $P^{-1}AP = D$, where D is the zero matrix.
 (ii) Find A in terms of P , P^{-1} , and D .
 (iii) Find a contradiction and conclude that nonzero nilpotent matrices are not diagonalizable.

47. **Proof** Prove that if A is a nonsingular diagonalizable matrix, then A^{-1} is also diagonalizable.

48. **CAPSTONE** Explain how to determine whether an $n \times n$ matrix A is diagonalizable using (a) similar matrices, (b) eigenvectors, and (c) distinct eigenvalues.

Showing That a Matrix Is Not Diagonalizable In Exercises 49 and 50, show that the matrix is not diagonalizable.

49. $\begin{bmatrix} 4 & k \\ 0 & 4 \end{bmatrix}, k \neq 0$ 50. $\begin{bmatrix} 0 & 0 \\ k & 0 \end{bmatrix}, k \neq 0$

7.3 Symmetric Matrices and Orthogonal Diagonalization

-  Recognize, and apply properties of, symmetric matrices.
-  Recognize, and apply properties of, orthogonal matrices.
-  Find an orthogonal matrix P that orthogonally diagonalizes a symmetric matrix A .

SYMMETRIC MATRICES

For most matrices, you must go through much of the diagonalization process before determining whether diagonalization is possible. One exception is with a triangular matrix that has distinct entries on the main diagonal. Such a matrix can be recognized as diagonalizable by inspection. In this section, you will study another type of matrix that is guaranteed to be diagonalizable: a **symmetric** matrix. Recall the definition below.

Definition of a Symmetric Matrix

A square matrix A is **symmetric** when it is equal to its transpose: $A = A^T$.

DISCOVERY

1. Pick an arbitrary nonsymmetric square matrix and calculate its eigenvalues.
2. Can you find a nonsymmetric square matrix for which the eigenvalues are not real?
3. Now pick an arbitrary symmetric matrix and calculate its eigenvalues.
4. Can you find a symmetric matrix for which the eigenvalues are not real?
5. What can you conclude about the eigenvalues of a symmetric matrix?

See LarsonLinearAlgebra.com for an interactive version of this type of exercise.

EXAMPLE 1

Symmetric Matrices and Nonsymmetric Matrices

The matrices A and B are symmetric, but the matrix C is not.

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -4 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Nonsymmetric matrices have properties that are not exhibited by symmetric matrices, as listed below.

1. A nonsymmetric matrix may not be diagonalizable.
2. A nonsymmetric matrix can have eigenvalues that are not real. For example, the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

has a characteristic equation of $\lambda^2 + 1 = 0$. So, its eigenvalues are the imaginary numbers $\lambda_1 = i$ and $\lambda_2 = -i$.

3. For a nonsymmetric matrix, the number of linearly independent eigenvectors corresponding to an eigenvalue can be less than the multiplicity of the eigenvalue. (See Example 6, Section 7.2.)

Theorem 7.7 lists properties of *symmetric* matrices.

THEOREM 7.7 Properties of Symmetric Matrices

If A is an $n \times n$ symmetric matrix, then the properties listed below are true.

1. A is diagonalizable.
2. All eigenvalues of A are real.
3. If λ is an eigenvalue of A with multiplicity k , then λ has k linearly independent eigenvectors. That is, the eigenspace of λ has dimension k .

REMARK

Theorem 7.7 is called the **Real Spectral Theorem**, and the set of eigenvalues of A is called the **spectrum** of A .

A proof of Theorem 7.7 is beyond the scope of this text. The next example proves that every 2×2 symmetric matrix is diagonalizable.

EXAMPLE 2**Every 2×2 Symmetric Matrix Is Diagonalizable**

Prove that a symmetric matrix

$$A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

is diagonalizable.

SOLUTION

The characteristic polynomial of A is

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - a & -c \\ -c & \lambda - b \end{vmatrix} \\ &= \lambda^2 - (a + b)\lambda + ab - c^2. \end{aligned}$$

As a quadratic in λ , this polynomial has a discriminant of

$$\begin{aligned} (a + b)^2 - 4(ab - c^2) &= a^2 + 2ab + b^2 - 4ab + 4c^2 \\ &= a^2 - 2ab + b^2 + 4c^2 \\ &= (a - b)^2 + 4c^2. \end{aligned}$$

This discriminant is the sum of two squares, so it must be either zero or positive. If $(a - b)^2 + 4c^2 = 0$, then $a = b$ and $c = 0$, so A is already diagonal. That is,

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

On the other hand, if $(a - b)^2 + 4c^2 > 0$, then by the Quadratic Formula the characteristic polynomial of A has two distinct real roots, which means that A has two distinct real eigenvalues. So, A is diagonalizable in this case as well. 

EXAMPLE 3**Dimensions of the Eigenspaces of a Symmetric Matrix**

Find the eigenvalues of the symmetric matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

and determine the dimensions of the corresponding eigenspaces.

SOLUTION

The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 2 & 0 & 0 \\ 2 & \lambda - 1 & 0 & 0 \\ 0 & 0 & \lambda - 1 & 2 \\ 0 & 0 & 2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)^2(\lambda - 3)^2.$$

So, the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 3$. Each of these eigenvalues has a multiplicity of 2, so you know from Theorem 7.7 that the corresponding eigenspaces also have dimension 2. Specifically, the eigenspace of $\lambda_1 = -1$ has a basis of $B_1 = \{(1, 1, 0, 0), (0, 0, 1, 1)\}$ and the eigenspace of $\lambda_2 = 3$ has a basis of $B_2 = \{(1, -1, 0, 0), (0, 0, 1, -1)\}$. (Verify these.) 

ORTHOGONAL MATRICES

To diagonalize a square matrix A , you need to find an *invertible* matrix P such that $P^{-1}AP$ is diagonal. For symmetric matrices, the matrix P can be chosen to have the special property that $P^{-1} = P^T$. This unusual matrix property is defined below.

Definition of an Orthogonal Matrix

A square matrix P is **orthogonal** when it is invertible and $P^{-1} = P^T$.

EXAMPLE 4

Orthogonal Matrices

a. The matrix $P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is orthogonal because $P^{-1} = P^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

b. The matrix

$$P = \begin{bmatrix} \frac{3}{5} & 0 & -\frac{4}{5} \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}$$

is orthogonal because

$$P^{-1} = P^T = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ 0 & 1 & 0 \\ -\frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}.$$

In Example 4, the columns of the matrices P form orthonormal sets in R^2 and R^3 , respectively (verify this), which suggests the next theorem.

THEOREM 7.8 Property of Orthogonal Matrices

An $n \times n$ matrix P is orthogonal if and only if its column vectors form an orthonormal set.

PROOF

To prove the theorem in one direction, assume that the column vectors of P form an orthonormal set:

$$P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n] \\ = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}.$$

Then the product P^TP has the form

$$P^TP = \begin{bmatrix} \mathbf{p}_1 \cdot \mathbf{p}_1 & \mathbf{p}_1 \cdot \mathbf{p}_2 & \cdots & \mathbf{p}_1 \cdot \mathbf{p}_n \\ \mathbf{p}_2 \cdot \mathbf{p}_1 & \mathbf{p}_2 \cdot \mathbf{p}_2 & \cdots & \mathbf{p}_2 \cdot \mathbf{p}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{p}_n \cdot \mathbf{p}_1 & \mathbf{p}_n \cdot \mathbf{p}_2 & \cdots & \mathbf{p}_n \cdot \mathbf{p}_n \end{bmatrix}.$$

The set

$$\{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\}$$

is orthonormal, so you have

$$\mathbf{p}_i \cdot \mathbf{p}_j = 0, i \neq j \quad \text{and} \quad \mathbf{p}_i \cdot \mathbf{p}_i = \|\mathbf{p}_i\|^2 = 1.$$

So, the matrix composed of dot products has the form

$$P^T P = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = I_n.$$

This implies that $P^T = P^{-1}$, so P is orthogonal.

Conversely, if P is orthogonal, then reverse the steps above to verify that the column vectors of P form an orthonormal set. 

EXAMPLE 5

An Orthogonal Matrix

Show that

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}$$

is orthogonal by showing that $P^T = P^{-1}$. Then show that the column vectors of P form an orthonormal set.

SOLUTION

$$P P^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix} = I_3$$

so it follows that $P^T = P^{-1}$. Moreover, letting

$$\mathbf{p}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{\sqrt{5}} \\ -\frac{2}{3\sqrt{5}} \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{\sqrt{5}} \\ -\frac{4}{3\sqrt{5}} \end{bmatrix}, \quad \text{and} \quad \mathbf{p}_3 = \begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{5}{3\sqrt{5}} \end{bmatrix}$$

produces

$$\mathbf{p}_1 \cdot \mathbf{p}_2 = \mathbf{p}_1 \cdot \mathbf{p}_3 = \mathbf{p}_2 \cdot \mathbf{p}_3 = 0$$

and

$$\|\mathbf{p}_1\| = \|\mathbf{p}_2\| = \|\mathbf{p}_3\| = 1.$$

So, $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is an orthonormal set, as guaranteed by Theorem 7.8. 

It can be shown that for a symmetric matrix, the eigenvectors corresponding to distinct eigenvalues are orthogonal. The next theorem states this property.

THEOREM 7.9 Property of Symmetric Matrices

Let A be an $n \times n$ symmetric matrix. If λ_1 and λ_2 are distinct eigenvalues of A , then their corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

PROOF

Let λ_1 and λ_2 be distinct eigenvalues of A with corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. So, $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$ and $A\mathbf{x}_2 = \lambda_2\mathbf{x}_2$. To prove the theorem, use the matrix form of the dot product, $\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_1^T \mathbf{x}_2$. (See Section 5.1.) Now you can write

$$\begin{aligned}
 \lambda_1(\mathbf{x}_1 \cdot \mathbf{x}_2) &= (\lambda_1 \mathbf{x}_1) \cdot \mathbf{x}_2 \\
 &= (A\mathbf{x}_1) \cdot \mathbf{x}_2 \\
 &= (A\mathbf{x}_1)^T \mathbf{x}_2 \\
 &= (\mathbf{x}_1^T A^T) \mathbf{x}_2 \\
 &= (\mathbf{x}_1^T A) \mathbf{x}_2 && A \text{ is symmetric, so } A = A^T. \\
 &= \mathbf{x}_1^T (A\mathbf{x}_2) \\
 &= \mathbf{x}_1^T (\lambda_2 \mathbf{x}_2) \\
 &= \mathbf{x}_1 \cdot (\lambda_2 \mathbf{x}_2) \\
 &= \lambda_2(\mathbf{x}_1 \cdot \mathbf{x}_2).
 \end{aligned}$$

This implies that $(\lambda_1 - \lambda_2)(\mathbf{x}_1 \cdot \mathbf{x}_2) = 0$. Also, $\lambda_1 \neq \lambda_2$, so it follows that $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$, which means that $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal. 

EXAMPLE 6 Eigenvectors of a Symmetric Matrix

Show that any two eigenvectors of

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

corresponding to distinct eigenvalues are orthogonal.

SOLUTION

The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{vmatrix} = (\lambda - 2)(\lambda - 4)$$

which implies that the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 4$. Verify that every eigenvector corresponding to $\lambda_1 = 2$ is of the form

$$\mathbf{x}_1 = \begin{bmatrix} s \\ -s \end{bmatrix}, \quad s \neq 0$$

and every eigenvector corresponding to $\lambda_2 = 4$ is of the form

$$\mathbf{x}_2 = \begin{bmatrix} t \\ t \end{bmatrix}, \quad t \neq 0.$$

So,

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = st - st = 0$$

which means that $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal. 

ORTHOGONAL DIAGONALIZATION

A matrix A is **orthogonally diagonalizable** when there exists an orthogonal matrix P such that $P^{-1}AP = D$ is diagonal. The important theorem below states that the set of orthogonally diagonalizable matrices is precisely the set of symmetric matrices.

THEOREM 7.10 Fundamental Theorem of Symmetric Matrices

Let A be an $n \times n$ matrix. Then A is orthogonally diagonalizable (and has real eigenvalues) if and only if A is symmetric.

PROOF

The proof of the theorem in one direction is fairly straightforward. That is, if you assume A is orthogonally diagonalizable, then there exists an orthogonal matrix P such that $D = P^{-1}AP$ is diagonal. Moreover, $P^{-1} = P^T$, so you have

$$\begin{aligned} A &= PDP^{-1} \\ &= PDP^T \end{aligned}$$

which implies that

$$\begin{aligned} A^T &= (PDP^T)^T \\ &= (P^T)^T D^T P^T \\ &= PDP^T \\ &= A. \end{aligned}$$

So, A is symmetric.

The proof of the theorem in the other direction is more involved, but it is important because it is constructive. Assume A is symmetric. If A has an eigenvalue λ of multiplicity k , then by Theorem 7.7, λ has k linearly independent eigenvectors. Through the Gram-Schmidt orthonormalization process, use this set of k vectors to form an orthonormal basis of eigenvectors for the eigenspace corresponding to λ . Repeat this procedure for each eigenvalue of A . The collection of all resulting eigenvectors is orthogonal by Theorem 7.9, and you know from the orthonormalization process that the collection is also orthonormal. Now let P be the matrix whose columns consist of these n orthonormal eigenvectors. By Theorem 7.8, P is an orthogonal matrix. Finally, by Theorem 7.5, $P^{-1}AP$ is diagonal. So, A is orthogonally diagonalizable. 

EXAMPLE 7

Determining Whether a Matrix Is Orthogonally Diagonalizable

Which matrices are orthogonally diagonalizable?

$$\begin{aligned} A_1 &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} & A_2 &= \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 8 \\ -1 & 8 & 0 \end{bmatrix} \\ A_3 &= \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} & A_4 &= \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \end{aligned}$$

SOLUTION

By Theorem 7.10, the orthogonally diagonalizable matrices are the symmetric ones: A_1 and A_4 . 

As mentioned above, the second part of the proof of Theorem 7.10 is *constructive*. That is, it gives you steps to follow to diagonalize a symmetric matrix orthogonally. A summary of these steps is on the next page.

Orthogonal Diagonalization of a Symmetric Matrix

Let A be an $n \times n$ symmetric matrix.

1. Find all eigenvalues of A and determine the multiplicity of each.
2. For *each* eigenvalue of multiplicity 1, find a unit eigenvector. (Find any eigenvector and then normalize it.)
3. For each eigenvalue of multiplicity $k \geq 2$, find a set of k linearly independent eigenvectors. (You know from Theorem 7.7 that this is possible.) If this set is not orthonormal, then apply the Gram-Schmidt orthonormalization process.
4. The results of Steps 2 and 3 produce an orthonormal set of n eigenvectors. Use these eigenvectors to form the columns of P . The matrix $P^{-1}AP = P^TAP = D$ will be diagonal. (The main diagonal entries of D are the eigenvalues of A .)

EXAMPLE 8

Orthogonal Diagonalization

Find a matrix P that orthogonally diagonalizes $A = \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$.

SOLUTION

1. The characteristic polynomial of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda + 2 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda + 3)(\lambda - 2).$$

So the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 2$.

2. For each eigenvalue, find an eigenvector by converting the matrix $\lambda I - A$ to reduced row-echelon form.

$$\begin{array}{lll} -3I - A = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} & \xrightarrow{\text{Eigenvector}} & \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ 2I - A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{array}$$

The eigenvectors $(-2, 1)$ and $(1, 2)$ form an *orthogonal* basis for R^2 . Normalize these eigenvectors to produce an *orthonormal* basis.

$$\mathbf{p}_1 = \frac{(-2, 1)}{\|(-2, 1)\|} = \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \quad \mathbf{p}_2 = \frac{(1, 2)}{\|(1, 2)\|} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

3. Each eigenvalue has a multiplicity of 1, so go directly to step 4.
4. Using $\mathbf{p}_1$ and $\mathbf{p}_2$ as column vectors, construct the matrix P .

$$P = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

Verify that P orthogonally diagonalizes A by finding $P^{-1}AP = P^TAP$.

$$P^TAP = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

EXAMPLE 9**Orthogonal Diagonalization**

See LarsonLinearAlgebra.com for an interactive version of this type of example.

Find a matrix P that orthogonally diagonalizes $A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}$.

SOLUTION

1. The characteristic polynomial of A , $|\lambda I - A| = (\lambda + 6)(\lambda - 3)^2$, yields the eigenvalues $\lambda_1 = -6$ and $\lambda_2 = 3$. The eigenvalue λ_1 has a multiplicity of 1 and the eigenvalue λ_2 has a multiplicity of 2.
2. An eigenvector for λ_1 is $\mathbf{v}_1 = (1, -2, 2)$, which normalizes to

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right).$$

3. Two eigenvectors for λ_2 are $\mathbf{v}_2 = (2, 1, 0)$ and $\mathbf{v}_3 = (-2, 0, 1)$. Note that $\mathbf{v}_1$ is orthogonal to $\mathbf{v}_2$ and $\mathbf{v}_3$ by Theorem 7.9. The eigenvectors $\mathbf{v}_2$ and $\mathbf{v}_3$, however, are not orthogonal to each other. To find two orthonormal eigenvectors for λ_2 , use the Gram-Schmidt process as shown below.

$$\mathbf{w}_2 = \mathbf{v}_2 = (2, 1, 0)$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \left(\frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\right)\mathbf{w}_2 = \left(-\frac{2}{5}, \frac{4}{5}, 1\right)$$

These vectors normalize to

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$$

$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \left(-\frac{2}{3\sqrt{5}}, \frac{4}{3\sqrt{5}}, \frac{5}{3\sqrt{5}}\right).$$

4. The matrix P has $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ as its column vectors.

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix}$$

$$\text{A check shows that } P^{-1}AP = P^TAP = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

**LINEAR ALGEBRA APPLIED**

The *Hessian matrix* is a symmetric matrix that can be helpful in finding relative maxima and minima of functions of several variables. For a function f of two variables x and y —that is, a surface in R^3 —the Hessian matrix has the form

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$$

The determinant of this matrix, evaluated at a point for which f_x and f_y are zero, is the expression used in the Second Partial Test for relative extrema.

Tacu Alexei/Shutterstock.com

7.3 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Determining Whether a Matrix Is Symmetric In Exercises 1 and 2, determine whether the matrix is symmetric.

$$1. \begin{bmatrix} 4 & -2 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \quad 2. \begin{bmatrix} 2 & 0 & 3 & 5 \\ 0 & 11 & 0 & -2 \\ 3 & 0 & 5 & 0 \\ 5 & -2 & 0 & 1 \end{bmatrix}$$

Proof In Exercises 3–6, prove that the symmetric matrix is diagonalizable.

$$3. A = \begin{bmatrix} 0 & 0 & a \\ 0 & a & 0 \\ a & 0 & 0 \end{bmatrix} \quad 4. A = \begin{bmatrix} 0 & a & 0 \\ a & 0 & a \\ 0 & a & 0 \end{bmatrix}$$

$$5. A = \begin{bmatrix} a & 0 & a \\ 0 & a & 0 \\ a & 0 & a \end{bmatrix} \quad 6. A = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}$$

Finding Eigenvalues and Dimensions of Eigenspaces In Exercises 7–18, find the eigenvalues of the symmetric matrix. For each eigenvalue, find the dimension of the corresponding eigenspace.

$$7. \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad 8. \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$9. \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad 10. \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$11. \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \quad 12. \begin{bmatrix} 0 & 4 & 4 \\ 4 & 2 & 0 \\ 4 & 0 & -2 \end{bmatrix}$$

$$13. \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad 14. \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$15. \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 5 & 3 \end{bmatrix} \quad 16. \begin{bmatrix} -1 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

$$17. \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$18. \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Determining Whether a Matrix Is Orthogonal In Exercises 19–32, determine whether the matrix is orthogonal. If the matrix is orthogonal, then show that the column vectors of the matrix form an orthonormal set.

$$19. \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \quad 20. \begin{bmatrix} \frac{4}{9} & -\frac{4}{9} \\ \frac{4}{9} & \frac{3}{9} \end{bmatrix}$$

$$21. \begin{bmatrix} -0.936 & -0.352 \\ 0.352 & -0.936 \end{bmatrix} \quad 22. \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$23. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad 24. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$25. \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \quad 26. \begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$$

$$27. \begin{bmatrix} -4 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{bmatrix} \quad 28. \begin{bmatrix} 4 & -1 & -4 \\ -1 & 0 & -17 \\ 1 & 4 & -1 \end{bmatrix}$$

$$29. \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{bmatrix}$$

$$30. \begin{bmatrix} \frac{\sqrt{2}}{3} & 0 & \frac{\sqrt{5}}{2} \\ 0 & \frac{2\sqrt{5}}{5} & 0 \\ -\frac{\sqrt{2}}{6} & -\frac{\sqrt{5}}{5} & \frac{1}{2} \end{bmatrix}$$

$$31. \begin{bmatrix} \frac{1}{8} & 0 & 0 & \frac{3}{8}\sqrt{7} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{3}{8}\sqrt{7} & 0 & 0 & \frac{1}{8} \end{bmatrix}$$

$$32. \begin{bmatrix} \frac{1}{10}\sqrt{10} & 0 & 0 & -\frac{3}{10}\sqrt{10} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{3}{10}\sqrt{10} & 0 & 0 & \frac{1}{10}\sqrt{10} \end{bmatrix}$$

Eigenvectors of a Symmetric Matrix In Exercises 33–38, show that any two eigenvectors of the symmetric matrix corresponding to distinct eigenvalues are orthogonal.

$$33. \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

$$34. \begin{bmatrix} -1 & -2 \\ -2 & 2 \end{bmatrix}$$

$$35. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$36. \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$37. \begin{bmatrix} 0 & \sqrt{3} & 0 \\ \sqrt{3} & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$38. \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Orthogonally Diagonalizable Matrices In Exercises 39–42, determine whether the matrix is orthogonally diagonalizable.

$$39. \begin{bmatrix} 4 & 5 \\ 0 & 1 \end{bmatrix}$$

$$40. \begin{bmatrix} 3 & 2 & -3 \\ -2 & -1 & 2 \\ -3 & 2 & 3 \end{bmatrix}$$

$$41. \begin{bmatrix} 5 & -3 & 8 \\ -3 & -3 & -3 \\ 8 & -3 & 8 \end{bmatrix}$$

$$42. \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{bmatrix}$$

Orthogonal Diagonalization In Exercises 43–52, find a matrix P such that P^TAP orthogonally diagonalizes A . Verify that P^TAP gives the correct diagonal form.

$$43. A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$44. A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$45. A = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$$

$$46. A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$47. A = \begin{bmatrix} 0 & 10 & 10 \\ 10 & 5 & 0 \\ 10 & 0 & -5 \end{bmatrix}$$

$$48. A = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

$$49. A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

$$50. A = \begin{bmatrix} -2 & 2 & 4 \\ 2 & -2 & 4 \\ 4 & 4 & 4 \end{bmatrix}$$

$$51. A = \begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$52. A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

True or False? In Exercises 53 and 54, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

53. (a) Let A be an $n \times n$ matrix. Then A is symmetric if and only if A is orthogonally diagonalizable.

(b) The eigenvectors corresponding to distinct eigenvalues are orthogonal for symmetric matrices.

54. (a) A square matrix P is orthogonal when it is invertible—that is, when $P^{-1} = P^T$.

(b) If A is an $n \times n$ symmetric matrix, then A has real eigenvalues.

55. **Proof** Prove that if A and B are $n \times n$ orthogonal matrices, then AB and BA are orthogonal.

56. **Proof** Prove that if a symmetric matrix A has only one eigenvalue λ , then $A = \lambda I$.

57. **Proof** Prove that if A is an orthogonal matrix, then so are A^T and A^{-1} .

58. **CAPSTONE** Consider the matrix below.

$$A = \begin{bmatrix} -1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 \end{bmatrix}$$

- Is A symmetric? Explain.
- Is A diagonalizable? Explain.
- Are the eigenvalues of A real? Explain.
- The eigenvalues of A are distinct. What are the dimensions of the corresponding eigenspaces? Explain.
- Is A orthogonal? Explain.
- For the eigenvalues of A , are the corresponding eigenvectors orthogonal? Explain.
- Is A orthogonally diagonalizable? Explain.

59. **Proof** Prove that the matrix below is orthogonal for any value of θ .

$$\begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

60. Find A^TA and AA^T for the matrix below. What do you observe?

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & -6 & 1 \end{bmatrix}$$

7.4 Applications of Eigenvalues and Eigenvectors

- Model population growth using an age transition matrix and an age distribution vector, and find a stable age distribution vector.
- Use a matrix equation to solve a system of first-order linear differential equations.
- Find the matrix of a quadratic form and use the Principal Axes Theorem to perform a rotation of axes for a conic and a quadric surface.
- Solve a constrained optimization problem.

POPULATION GROWTH

Matrices can be used to form models for population growth. The first step in this process is to group the population into age classes of equal duration. For example, if the maximum life span of a member is M years, then the n intervals below represent the age classes.

$$\begin{array}{ll} \left[0, \frac{M}{n}\right) & \text{First age class} \\ \left[\frac{M}{n}, \frac{2M}{n}\right) & \text{Second age class} \\ \vdots & \vdots \\ \left[\frac{(n-1)M}{n}, M\right] & \text{\textit{n}th age class} \end{array}$$

The **age distribution vector** $\mathbf{x}$ represents the number of population members in each age class, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \begin{array}{l} \text{Number in first age class} \\ \text{Number in second age class} \\ \vdots \\ \text{Number in \textit{n}th age class} \end{array}$$

Over a period of M/n years, the *probability* that a member of the i th age class will survive to become a member of the $(i + 1)$ th age class is p_i , where

$$0 \leq p_i \leq 1, i = 1, 2, \dots, n - 1.$$

The *average number* of offspring produced by a member of the i th age class is b_i , where $0 \leq b_i, i = 1, 2, \dots, n$. These numbers can be written in matrix form, as shown below.

$$L = \begin{bmatrix} b_1 & b_2 & \dots & b_{n-1} & b_n \\ p_1 & 0 & \dots & 0 & 0 \\ 0 & p_2 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & p_{n-1} & 0 \end{bmatrix}$$

REMARK

Recall from Section 6.4 that the age transition matrix L is also called a **Leslie matrix** after mathematician Patrick H. Leslie.

Multiplying this **age transition matrix** by the age distribution vector for a specific time period produces the age distribution vector for the next time period. That is,

$$L\mathbf{x}_j = \mathbf{x}_{j+1}.$$

Example 1 illustrates this procedure.

REMARK

In Example 1, verify that the rabbit population after 2 years is

$$\mathbf{x}_3 = L\mathbf{x}_2 = \begin{bmatrix} 168 \\ 152 \\ 6 \end{bmatrix}.$$

Notice from the age distribution vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ that the percent of rabbits in each of the three age classes changes each year. To obtain a stable growth pattern, one in which the percent in each age class remains the same each year, the $(n + 1)$ th age distribution vector must be a scalar multiple of the n th age distribution vector. That is, $\mathbf{x}_{n+1} = L\mathbf{x}_n = \lambda\mathbf{x}_n$. Example 2 shows how to solve this problem.

EXAMPLE 1**A Population Growth Model**

A population of rabbits has the characteristics below.

a. Half of the rabbits survive their first year. Of those, half survive their second year. The maximum life span is 3 years.

b. During the first year, the rabbits produce no offspring. The average number of offspring is 6 during the second year and 8 during the third year.

The population now consists of 24 rabbits in the first age class, 24 in the second, and 20 in the third. How many rabbits will there be in each age class in 1 year?

SOLUTION

The current age distribution vector is

$$\mathbf{x}_1 = \begin{bmatrix} 24 \\ 24 \\ 20 \end{bmatrix} \quad \begin{array}{l} 0 \leq \text{age} < 1 \\ 1 \leq \text{age} < 2 \\ 2 \leq \text{age} \leq 3 \end{array}$$

and the age transition matrix is

$$L = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}.$$

After 1 year, the age distribution vector will be

$$\mathbf{x}_2 = L\mathbf{x}_1 = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 24 \\ 24 \\ 20 \end{bmatrix} = \begin{bmatrix} 304 \\ 12 \\ 12 \end{bmatrix}. \quad \begin{array}{l} 0 \leq \text{age} < 1 \\ 1 \leq \text{age} < 2 \\ 2 \leq \text{age} \leq 3 \end{array}$$

EXAMPLE 2**Finding a Stable Age Distribution Vector**

Find a stable age distribution vector for the population in Example 1.

SOLUTION

To solve this problem, find an eigenvalue λ and a corresponding eigenvector $\mathbf{x}$ such that $L\mathbf{x} = \lambda\mathbf{x}$. The characteristic polynomial of L is

$$|\lambda I - L| = (\lambda + 1)^2(\lambda - 2)$$

(check this), which implies that the eigenvalues are -1 and 2 . Choosing the positive value, let $\lambda = 2$. Verify that the corresponding eigenvectors are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16t \\ 4t \\ t \end{bmatrix} = t \begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix}.$$

For example, if $t = 2$, then the initial age distribution vector is

$$\mathbf{x}_1 = \begin{bmatrix} 32 \\ 8 \\ 2 \end{bmatrix} \quad \begin{array}{l} 0 \leq \text{age} < 1 \\ 1 \leq \text{age} < 2 \\ 2 \leq \text{age} \leq 3 \end{array}$$

and the age distribution vector for the next year is

$$\mathbf{x}_2 = L\mathbf{x}_1 = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 32 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} 64 \\ 16 \\ 4 \end{bmatrix}. \quad \begin{array}{l} 0 \leq \text{age} < 1 \\ 1 \leq \text{age} < 2 \\ 2 \leq \text{age} \leq 3 \end{array}$$

Notice that the ratio of the three age classes is still $16 : 4 : 1$, and so the percent of the population in each age class remains the same.

SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS (CALCULUS)

A system of first-order linear differential equations has the form

$$\begin{aligned}y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\&\vdots \\y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n\end{aligned}$$

where each y_i is a function of t and $y_i' = \frac{dy_i}{dt}$. If you let

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

then the system can be written in matrix form as

$$\mathbf{y}' = A\mathbf{y}.$$

EXAMPLE 3

Solving a System of Linear Differential Equations

Solve the system of linear differential equations.

$$\begin{aligned}y_1' &= 4y_1 \\y_2' &= -y_2 \\y_3' &= 2y_3\end{aligned}$$

SOLUTION

From calculus, you know that the solution of the differential equation $y' = ky$ is

$$y = Ce^{kt}.$$

So, the solution of the system is

$$\begin{aligned}y_1 &= C_1 e^{4t} \\y_2 &= C_2 e^{-t} \\y_3 &= C_3 e^{2t}.\end{aligned}$$

The matrix form of the system of linear differential equations in Example 3 is $\mathbf{y}' = A\mathbf{y}$, or

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

So, the coefficients of t in the solutions $y_i = C_i e^{\lambda_i t}$ are the *eigenvalues* of the matrix A .

If A is a *diagonal* matrix, then the solution of

$$\mathbf{y}' = A\mathbf{y}$$

can be obtained immediately, as in Example 3. If A is *not* diagonal, then the solution requires more work. First, find a matrix P that diagonalizes A . Then, the change of variables $\mathbf{y} = P\mathbf{w}$ and $\mathbf{y}' = P\mathbf{w}'$ produces

$$P\mathbf{w}' = \mathbf{y}' = A\mathbf{y} = AP\mathbf{w} \quad \rightarrow \quad \mathbf{w}' = P^{-1}AP\mathbf{w}$$

where $P^{-1}AP$ is a diagonal matrix. Example 4 demonstrates this procedure.

EXAMPLE 4**Solving a System of Linear Differential Equations**

See LarsonLinearAlgebra.com for an interactive version of this type of example.

To solve the system of linear differential equations

$$y_1' = 3y_1 + 2y_2$$

$$y_2' = 6y_1 - y_2$$

first find a matrix P that diagonalizes $A = \begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix}$. Verify that the eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = 5$, and that the corresponding eigenvectors are $\mathbf{p}_1 = [1 \ -3]^T$ and $\mathbf{p}_2 = [1 \ 1]^T$. Diagonalize A using the matrix P whose columns consist of $\mathbf{p}_1$ and $\mathbf{p}_2$ to obtain

$$P = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}, \quad \text{and} \quad P^{-1}AP = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}.$$

The system $\mathbf{w}' = P^{-1}AP\mathbf{w}$ has the form below.

$$\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \rightarrow \quad \begin{matrix} w_1' = -3w_1 \\ w_2' = 5w_2 \end{matrix}$$

The solution of this system of equations is

$$w_1 = C_1 e^{-3t}$$

$$w_2 = C_2 e^{5t}.$$

To return to the original variables y_1 and y_2 , use the substitution $\mathbf{y} = P\mathbf{w}$ and write

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

which implies that the solution is

$$y_1 = w_1 + w_2 = C_1 e^{-3t} + C_2 e^{5t}$$

$$y_2 = -3w_1 + w_2 = -3C_1 e^{-3t} + C_2 e^{5t}.$$

If A has eigenvalues with multiplicity greater than 1 or if A has complex eigenvalues, then the technique for solving the system must be modified.

1. Eigenvalues with multiplicity greater than 1: The coefficient matrix of the system

$$\begin{matrix} y_1' = & y_2 \\ y_2' = -4y_1 + 4y_2 \end{matrix} \quad \text{is} \quad A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}.$$

The only eigenvalue of A is $\lambda = 2$, and the solution of the system is

$$y_1 = C_1 e^{2t} + C_2 t e^{2t}$$

$$y_2 = (2C_1 + C_2) e^{2t} + 2C_2 t e^{2t}.$$

2. Complex eigenvalues: The coefficient matrix of the system

$$\begin{matrix} y_1' = -y_2 \\ y_2' = y_1 \end{matrix} \quad \text{is} \quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = i$ and $\lambda_2 = -i$, and the solution of the system is

$$y_1 = C_1 \cos t + C_2 \sin t$$

$$y_2 = -C_2 \cos t + C_1 \sin t.$$

Check these solutions by differentiating and substituting into the original systems of equations.

QUADRATIC FORMS

Eigenvalues and eigenvectors can be used to solve the rotation of axes problem introduced in Section 4.8. Recall that classifying the graph of the quadratic equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \quad \text{Quadratic equation}$$

is fairly straightforward as long as the equation has no xy -term (that is, $b = 0$). If the equation has an xy -term, however, then the classification is accomplished most easily by first performing a rotation of axes that eliminates the xy -term. The resulting equation (relative to the new $x'y'$ -axes) will then be of the form

$$a'(x')^2 + c'(y')^2 + d'x' + e'y' + f' = 0.$$

You will see that the coefficients a' and c' are eigenvalues of the matrix

$$A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}.$$

The expression

$$ax^2 + bxy + cy^2 \quad \text{Quadratic form}$$

is the **quadratic form** associated with the quadratic equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

and the matrix A is the **matrix of the quadratic form**. Note that the matrix A is *symmetric*. Moreover, the matrix A will be diagonal if and only if its corresponding quadratic form has no xy -term, as illustrated in Example 5.

EXAMPLE 5

Finding the Matrix of the Quadratic Form

Find the matrix of the quadratic form associated with each quadratic equation.

a. $4x^2 + 9y^2 - 36 = 0$ b. $13x^2 - 10xy + 13y^2 - 72 = 0$

SOLUTION

a. $a = 4$, $b = 0$, and $c = 9$, so the matrix is

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}. \quad \text{Diagonal matrix (no } xy\text{-term)}$$

b. $a = 13$, $b = -10$, and $c = 13$, so the matrix is

$$A = \begin{bmatrix} 13 & -5 \\ -5 & 13 \end{bmatrix}. \quad \text{Nondiagonal matrix (} xy\text{-term)}$$

In standard form, the equation $4x^2 + 9y^2 - 36 = 0$ is

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

which is the equation of the ellipse shown in Figure 7.2. Although it is not apparent by inspection, the graph of the equation $13x^2 - 10xy + 13y^2 - 72 = 0$ is similar. In fact, when you rotate the x - and y -axes counterclockwise 45° to form a new $x'y'$ -coordinate system, this equation takes the form

$$\frac{(x')^2}{3^2} + \frac{(y')^2}{2^2} = 1$$

(verify this) which is the equation of the ellipse shown in Figure 7.3.

To see how to use the matrix of a quadratic form to perform a rotation of axes, let

$$X = \begin{bmatrix} x & y \end{bmatrix}^T.$$

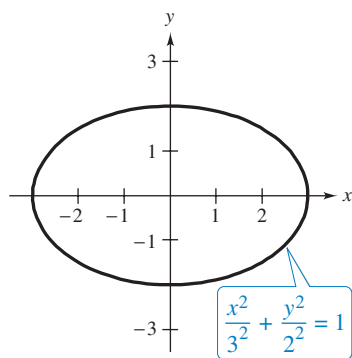


Figure 7.2

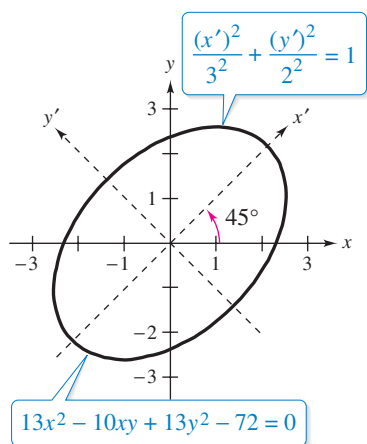


Figure 7.3

REMARK

For brevity, the 1×1 matrices $[f]$ and $[ax^2 + bxy + cy^2 + ey + f]$ are not shown enclosed in brackets.

Then the quadratic expression $ax^2 + bxy + cy^2 + dx + ey + f$ can be written in matrix form as shown below.

$$\begin{aligned} X^TAX + [d \ e]X + f &= [x \ y] \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + [d \ e] \begin{bmatrix} x \\ y \end{bmatrix} + f \\ &= ax^2 + bxy + cy^2 + dx + ey + f \end{aligned}$$

If $b = 0$, then no rotation is necessary. But if $b \neq 0$, then use the fact that A is symmetric and apply Theorem 7.10 to conclude that there exists an orthogonal matrix P such that $P^TAP = D$ is diagonal. So, if you let

$$P^TX = X' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

then it follows that $X = PX'$, and $X^TAX = (PX')^T A(PX') = (X')^T P^TAPX' = (X')^TDX'$.

The choice of the matrix P must be made with care. P is orthogonal, so its determinant will be ± 1 . It can be shown (see Exercise 67) that if P is chosen so that $|P| = 1$, then P will be of the form

$$P = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

where θ is the angle of rotation of the conic measured from the positive x -axis to the positive x' -axis. This leads to the **Principal Axes Theorem**.

Principal Axes Theorem

For a conic whose equation is $ax^2 + bxy + cy^2 + dx + ey + f = 0$, the rotation $X = PX'$ eliminates the xy -term when P is an orthogonal matrix, with $|P| = 1$, that diagonalizes the matrix of the quadratic form A . That is,

$$P^TAP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where λ_1 and λ_2 are eigenvalues of A . The equation of the rotated conic is

$$\lambda_1(x')^2 + \lambda_2(y')^2 + [d \ e]PX' + f = 0.$$

REMARK

Note that the matrix product $[d \ e]PX'$ has the form

$$(d \cos \theta + e \sin \theta)x' + (-d \sin \theta + e \cos \theta)y'.$$

EXAMPLE 6**Rotation of a Conic**

Perform a rotation of axes to eliminate the xy -term in the quadratic equation

$$13x^2 - 10xy + 13y^2 - 72 = 0.$$

SOLUTION

The matrix of the quadratic form associated with this equation is

$$A = \begin{bmatrix} 13 & -5 \\ -5 & 13 \end{bmatrix}.$$

The characteristic polynomial of A is $(\lambda - 8)(\lambda - 18)$ (check this), so it follows that the eigenvalues of A are $\lambda_1 = 8$ and $\lambda_2 = 18$. Then, the equation of the rotated conic is

$$8(x')^2 + 18(y')^2 - 72 = 0$$

which, when written in the standard form

$$\frac{(x')^2}{3^2} + \frac{(y')^2}{2^2} = 1$$

is the equation of an ellipse. (See Figure 7.3.)

In Example 6, the eigenvectors of the matrix A are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

which you can normalize to form the columns of P , as shown below.

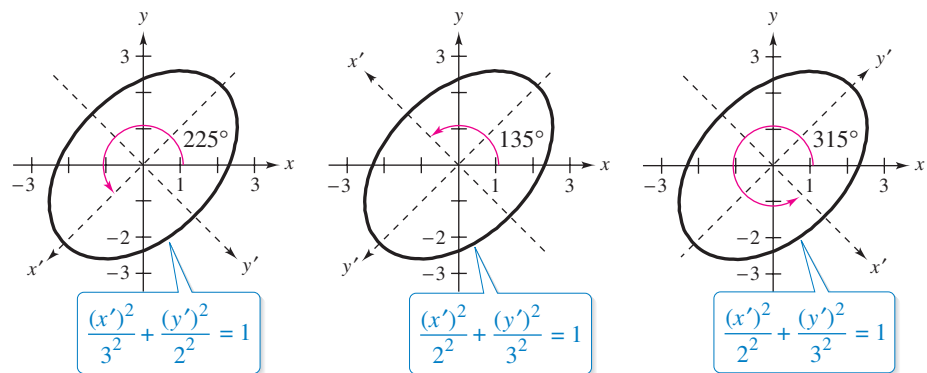
$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Note first that $|P| = 1$, which implies that P is a rotation. Moreover, $\cos 45^\circ = 1/\sqrt{2} = \sin 45^\circ$, so the angle of rotation is 45° as shown in Figure 7.3.

The orthogonal matrix P specified in the Principal Axes Theorem is not unique. Its entries depend on the ordering of the eigenvalues λ_1 and λ_2 and on the subsequent choice of eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. For instance, in the solution of Example 6, any of the choices of P shown below would have worked.

$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_1$	$\mathbf{x}_2$
$\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$
$\lambda_1 = 8, \lambda_2 = 18$		$\lambda_1 = 18, \lambda_2 = 8$		$\lambda_1 = 18, \lambda_2 = 8$	
$\theta = 225^\circ$		$\theta = 135^\circ$		$\theta = 315^\circ$	

For any of these choices of P , the graph of the rotated conic will, of course, be the same. (See below.)



The list below summarizes the steps used to apply the Principal Axes Theorem.

1. Form the matrix A and find its eigenvalues λ_1 and λ_2 .
2. Find eigenvectors corresponding to λ_1 and λ_2 . Normalize these eigenvectors to form the columns of P .
3. If $|P| = -1$, then multiply one of the columns of P by -1 to obtain a matrix of the form

$$P = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

4. The angle θ represents the angle of rotation of the conic.
5. The equation of the rotated conic is $\lambda_1(x')^2 + \lambda_2(y')^2 + [d \ e]PX' + f = 0$.

Example 7 shows how to apply the Principal Axes Theorem to rotate a conic whose center is translated away from the origin.

EXAMPLE 7**Rotation of a Conic**

Perform a rotation of axes to eliminate the xy -term in the quadratic equation

$$3x^2 - 10xy + 3y^2 + 16\sqrt{2}x - 32 = 0.$$

SOLUTION

The matrix of the quadratic form associated with this equation is

$$A = \begin{bmatrix} 3 & -5 \\ -5 & 3 \end{bmatrix}.$$

The eigenvalues of A are

$$\lambda_1 = 8 \quad \text{and} \quad \lambda_2 = -2$$

with corresponding eigenvectors of

$$\mathbf{x}_1 = (-1, 1) \quad \text{and} \quad \mathbf{x}_2 = (-1, -1).$$

This implies that the matrix P is

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \text{ where } |P| = 1.$$

$\cos 135^\circ = -1/\sqrt{2}$ and $\sin 135^\circ = 1/\sqrt{2}$, so the angle of rotation is 135° . Finally, from the matrix product

$$\begin{aligned} [d \quad e]PX' &= [16\sqrt{2} \quad 0] \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \\ &= -16x' - 16y' \end{aligned}$$

the equation of the rotated conic is

$$8(x')^2 - 2(y')^2 - 16x' - 16y' - 32 = 0.$$

In standard form, the equation is

$$\frac{(x' - 1)^2}{1^2} - \frac{(y' + 4)^2}{2^2} = 1$$

which is the equation of a hyperbola. Its graph is shown in Figure 7.4.

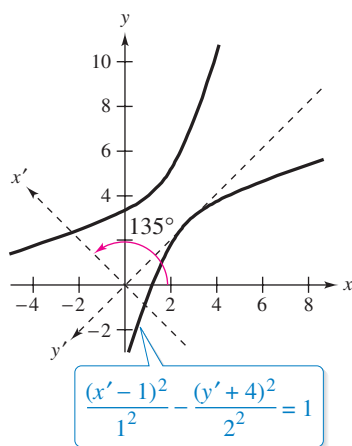
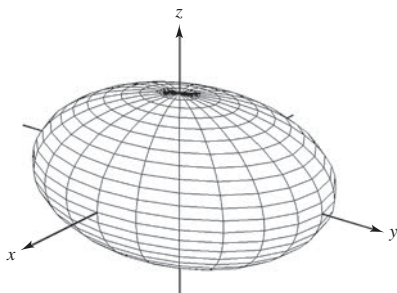
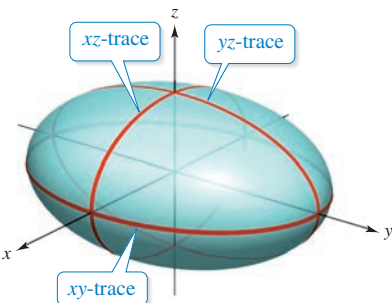
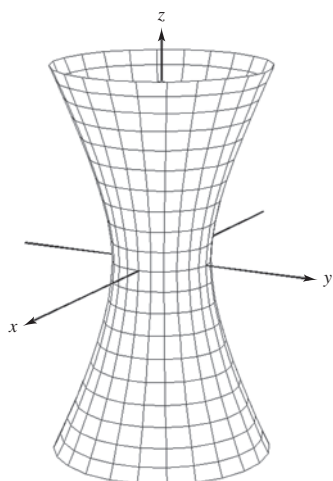
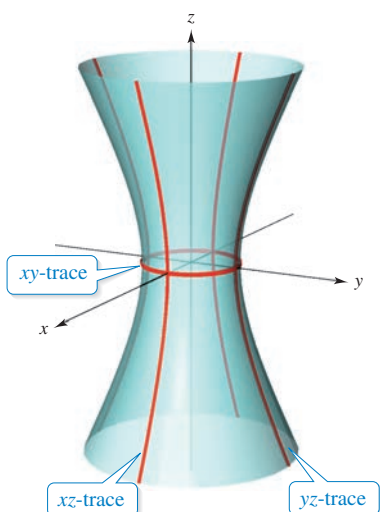
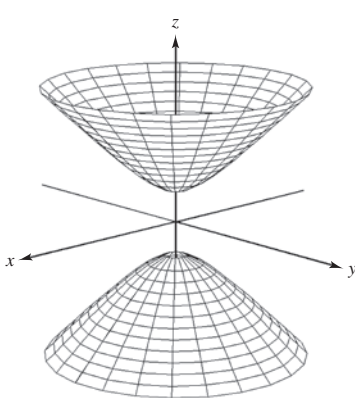
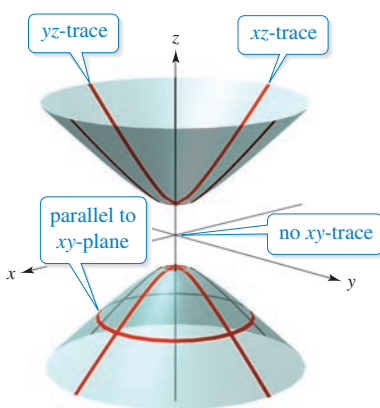


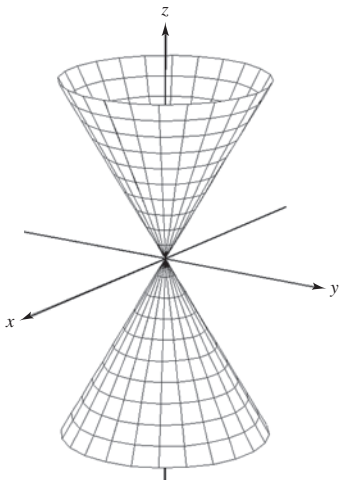
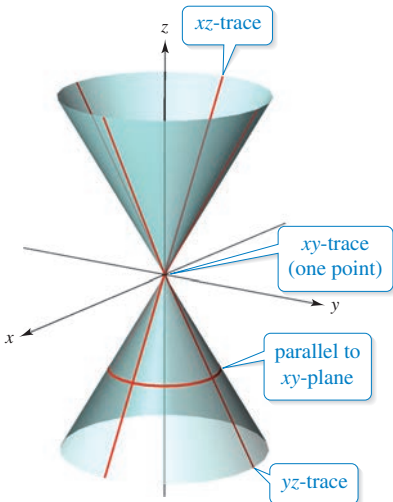
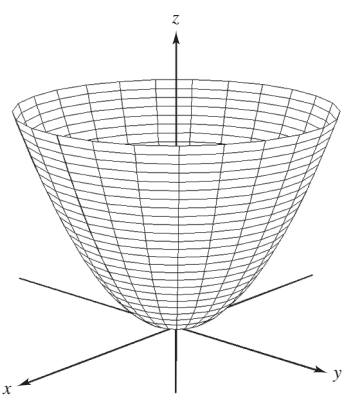
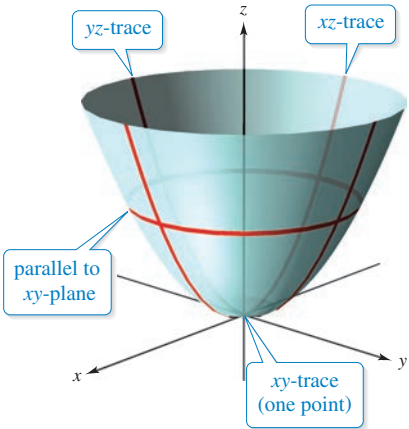
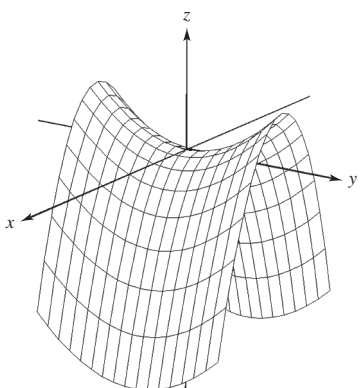
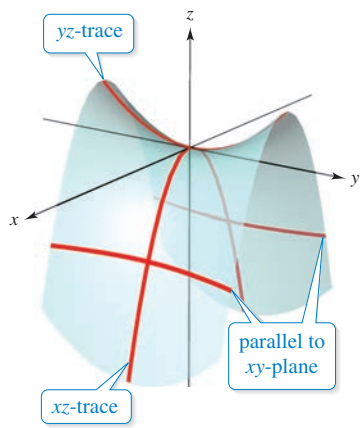
Figure 7.4

Quadratic forms can also be used to analyze equations of quadric surfaces in R^3 , which are the three-dimensional analogs of conic sections. The equation of a quadric surface in R^3 is a second-degree polynomial of the form

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0.$$

There are six basic types of quadric surfaces: ellipsoids, hyperboloids of one sheet, hyperboloids of two sheets, elliptic cones, elliptic paraboloids, and hyperbolic paraboloids. The intersection of a surface with a plane, called the **trace** of the surface in the plane, is useful to help visualize the graph of the surface in R^3 . The six basic types of quadric surfaces, together with their traces, are shown on the next two pages.

	<i>Ellipsoid</i> $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ <table><tr><td><u>Trace</u></td><td><u>Plane</u></td></tr><tr><td>Ellipse</td><td>Parallel to xy-plane</td></tr><tr><td>Ellipse</td><td>Parallel to xz-plane</td></tr><tr><td>Ellipse</td><td>Parallel to yz-plane</td></tr></table> The surface is a sphere when $a = b = c \neq 0$.	<u>Trace</u>	<u>Plane</u>	Ellipse	Parallel to xy -plane	Ellipse	Parallel to xz -plane	Ellipse	Parallel to yz -plane	
<u>Trace</u>	<u>Plane</u>									
Ellipse	Parallel to xy -plane									
Ellipse	Parallel to xz -plane									
Ellipse	Parallel to yz -plane									
	<i>Hyperboloid of One Sheet</i> $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ <table><tr><td><u>Trace</u></td><td><u>Plane</u></td></tr><tr><td>Ellipse</td><td>Parallel to xy-plane</td></tr><tr><td>Hyperbola</td><td>Parallel to xz-plane</td></tr><tr><td>Hyperbola</td><td>Parallel to yz-plane</td></tr></table> The axis of the hyperboloid corresponds to the variable whose coefficient is negative.	<u>Trace</u>	<u>Plane</u>	Ellipse	Parallel to xy -plane	Hyperbola	Parallel to xz -plane	Hyperbola	Parallel to yz -plane	
<u>Trace</u>	<u>Plane</u>									
Ellipse	Parallel to xy -plane									
Hyperbola	Parallel to xz -plane									
Hyperbola	Parallel to yz -plane									
	<i>Hyperboloid of Two Sheets</i> $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ <table><tr><td><u>Trace</u></td><td><u>Plane</u></td></tr><tr><td>Ellipse</td><td>Parallel to xy-plane</td></tr><tr><td>Hyperbola</td><td>Parallel to xz-plane</td></tr><tr><td>Hyperbola</td><td>Parallel to yz-plane</td></tr></table> The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.	<u>Trace</u>	<u>Plane</u>	Ellipse	Parallel to xy -plane	Hyperbola	Parallel to xz -plane	Hyperbola	Parallel to yz -plane	
<u>Trace</u>	<u>Plane</u>									
Ellipse	Parallel to xy -plane									
Hyperbola	Parallel to xz -plane									
Hyperbola	Parallel to yz -plane									

	<i>Elliptic Cone</i> $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ <table><tr><th><u>Trace</u></th><th><u>Plane</u></th></tr><tr><td>Ellipse</td><td>Parallel to xy-plane</td></tr><tr><td>Hyperbola</td><td>Parallel to xz-plane</td></tr><tr><td>Hyperbola</td><td>Parallel to yz-plane</td></tr></table> The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.	<u>Trace</u>	<u>Plane</u>	Ellipse	Parallel to xy -plane	Hyperbola	Parallel to xz -plane	Hyperbola	Parallel to yz -plane	
<u>Trace</u>	<u>Plane</u>									
Ellipse	Parallel to xy -plane									
Hyperbola	Parallel to xz -plane									
Hyperbola	Parallel to yz -plane									
	<i>Elliptic Paraboloid</i> $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ <table><tr><th><u>Trace</u></th><th><u>Plane</u></th></tr><tr><td>Ellipse</td><td>Parallel to xy-plane</td></tr><tr><td>Parabola</td><td>Parallel to xz-plane</td></tr><tr><td>Parabola</td><td>Parallel to yz-plane</td></tr></table> The axis of the paraboloid corresponds to the variable raised to the first power.	<u>Trace</u>	<u>Plane</u>	Ellipse	Parallel to xy -plane	Parabola	Parallel to xz -plane	Parabola	Parallel to yz -plane	
<u>Trace</u>	<u>Plane</u>									
Ellipse	Parallel to xy -plane									
Parabola	Parallel to xz -plane									
Parabola	Parallel to yz -plane									
	<i>Hyperbolic Paraboloid</i> $z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$ <table><tr><th><u>Trace</u></th><th><u>Plane</u></th></tr><tr><td>Hyperbola</td><td>Parallel to xy-plane</td></tr><tr><td>Parabola</td><td>Parallel to xz-plane</td></tr><tr><td>Parabola</td><td>Parallel to yz-plane</td></tr></table> The axis of the paraboloid corresponds to the variable raised to the first power.	<u>Trace</u>	<u>Plane</u>	Hyperbola	Parallel to xy -plane	Parabola	Parallel to xz -plane	Parabola	Parallel to yz -plane	
<u>Trace</u>	<u>Plane</u>									
Hyperbola	Parallel to xy -plane									
Parabola	Parallel to xz -plane									
Parabola	Parallel to yz -plane									

The quadratic form of the equation

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0 \quad \text{Quadric surface}$$

is

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz. \quad \text{Quadratic form}$$

The corresponding matrix is

$$A = \begin{bmatrix} a & \frac{d}{2} & \frac{e}{2} \\ \frac{d}{2} & b & \frac{f}{2} \\ \frac{e}{2} & \frac{f}{2} & c \end{bmatrix}.$$

In its three-dimensional version, the Principal Axes Theorem relates the eigenvalues and eigenvectors of A to the equation of the rotated surface, as shown in Example 8.

REMARK

In general, the matrix A of the quadratic form will always be symmetric.

EXAMPLE 8 Rotation of a Quadric Surface

Perform a rotation of axes to eliminate the xz -term in the quadratic equation

$$5x^2 + 4y^2 + 5z^2 + 8xz - 36 = 0.$$

SOLUTION

The matrix A associated with this quadratic equation is

$$A = \begin{bmatrix} 5 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 5 \end{bmatrix}$$

which has eigenvalues of $\lambda_1 = 1$, $\lambda_2 = 4$, and $\lambda_3 = 9$ (verify this). So, in the rotated $x'y'z'$ -system, the quadratic equation is $(x')^2 + 4(y')^2 + 9(z')^2 - 36 = 0$, which in standard form is

$$\frac{(x')^2}{6^2} + \frac{(y')^2}{3^2} + \frac{(z')^2}{2^2} = 1.$$

The graph of this equation is an ellipsoid. As shown in Figure 7.5, the $x'y'z'$ -axes represent a counterclockwise rotation of 45° about the y -axis. Verify that the columns of

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

are the normalized eigenvectors of A , that P is orthogonal, and that P^TAP is diagonal.

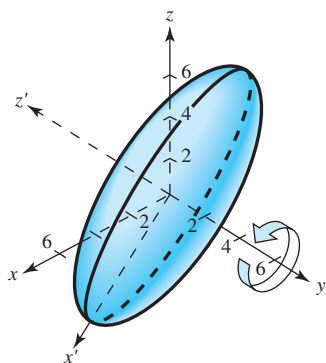


Figure 7.5



LINEAR ALGEBRA APPLIED

Some of the world's most unusual architecture makes use of quadric surfaces. For example, *Catedral Metropolitana Nossa Senhora Aparecida*, a cathedral located in Brasilia, Brazil, is in the shape of a hyperboloid of one sheet. It was designed by Pritzker Prize winning architect Oscar Niemeyer, and dedicated in 1970. The sixteen identical curved steel columns are intended to represent two hands reaching up to the sky. In the triangular gaps formed by the columns, semitransparent stained glass allows light inside for nearly the entire height of the columns.

ostill/Shutterstock.com

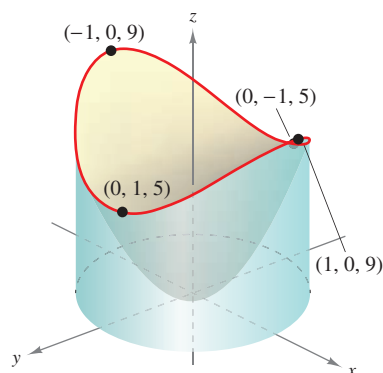


Figure 7.6

CONSTRAINED OPTIMIZATION

Many real-life applications require you to determine the maximum or minimum value of a quantity subject to a *constraint*. For instance, consider a simplified example in which you need to find the maximum and minimum values of the quadric surface $f(x, y) = 9x^2 + 5y^2$ along the curve formed by intersection of the surface with the unit cylinder $x^2 + y^2 = 1$, as shown in Figure 7.6. The constraint is the unit cylinder $x^2 + y^2 = 1$. By inspection, the maximum value of f is 9 when $x = \pm 1$ and $y = 0$, and the minimum value of f is 5 when $x = 0$ and $y = \pm 1$.

The theorem below allows you to use the eigenvalues and eigenvectors of a symmetric matrix to solve a constrained optimization problem.

Constrained Optimization Theorem

For a quadratic form f in n variables with matrix of the quadratic form A subject to the constraint $\|\mathbf{x}\|^2 = 1$, the maximum value of f is the greatest eigenvalue of A and the minimum value of f is the least eigenvalue of A .

PROOF

The quadratic form f can be written as

$$f(x_1, x_2, \dots, x_n) = \mathbf{x}^T A \mathbf{x}.$$

The matrix of the quadratic form A is symmetric, so A has n real eigenvalues (counting multiplicities). Call these $\lambda_1, \lambda_2, \dots, \lambda_n$, and assume that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. Now consider a change of variables $\mathbf{x} = P\mathbf{x}'$, where $\mathbf{x}' = [x_1' \ x_2' \ \dots \ x_n']^T$ and P is an orthogonal matrix that diagonalizes A . Then

$$\begin{aligned} f(x_1, x_2, \dots, x_n) &= \mathbf{x}^T A \mathbf{x} \\ &= (P\mathbf{x}')^T A (P\mathbf{x}') \\ &= (\mathbf{x}')^T P^T A P \mathbf{x}' \\ &= \lambda_1(x_1')^2 + \lambda_2(x_2')^2 + \dots + \lambda_n(x_n')^2 \end{aligned}$$

and

$$\begin{aligned} \|\mathbf{x}\|^2 &= \|P\mathbf{x}'\|^2 \\ &= (P\mathbf{x}')^T (P\mathbf{x}') \\ &= (\mathbf{x}')^T P^T P \mathbf{x}' \\ &= (\mathbf{x}')^T \mathbf{x}' \\ &= (x_1')^2 + (x_2')^2 + \dots + (x_n')^2 \\ &= \|\mathbf{x}'\|^2. \end{aligned}$$

$\|\mathbf{x}\|^2 = 1$, so $\|\mathbf{x}'\|^2 = 1$, and

$$\begin{aligned} \lambda_1 &= \lambda_1[(x_1')^2 + (x_2')^2 + \dots + (x_n')^2] \\ &\geq \lambda_1(x_1')^2 + \lambda_2(x_2')^2 + \dots + \lambda_n(x_n')^2 \\ &\geq \lambda_n[(x_1')^2 + (x_2')^2 + \dots + (x_n')^2] \\ &= \lambda_n. \end{aligned}$$

This shows that $\lambda_1 \geq \mathbf{x}^T A \mathbf{x} \geq \lambda_n$. So, all values of $f(x_1, x_2, \dots, x_n) = \mathbf{x}^T A \mathbf{x}$ for which $\|\mathbf{x}\|^2 = 1$ lie between λ_1 and λ_n . If $\mathbf{x}$ is a normalized eigenvector that corresponds to λ_1 , then

$$f(x_1, x_2, \dots, x_n) = \mathbf{x}^T A \mathbf{x} = \mathbf{x}^T (\lambda_1 \mathbf{x}) = \lambda_1 \|\mathbf{x}\|^2 = \lambda_1.$$

If $\mathbf{x}$ is a normalized eigenvector that corresponds to λ_n , then

$$f(x_1, x_2, \dots, x_n) = \mathbf{x}^T A \mathbf{x} = \mathbf{x}^T (\lambda_n \mathbf{x}) = \lambda_n \|\mathbf{x}\|^2 = \lambda_n.$$

So, f has a constrained maximum of λ_1 and a constrained minimum of λ_n .

EXAMPLE 9**Finding Maximum and Minimum Values**

Find the maximum and minimum values of $f(x_1, x_2) = 9x_1^2 + 5x_2^2$ subject to the constraint $\|\mathbf{x}\|^2 = 1$.

SOLUTION

The matrix of the quadratic form is the diagonal matrix

$$A = \begin{bmatrix} 9 & 0 \\ 0 & 5 \end{bmatrix}.$$

By inspection, the eigenvalues of A are $\lambda_1 = 9$ and $\lambda_2 = 5$. So by the Constrained Optimization Theorem, the maximum value of z is 9 and the minimum value of z is 5.

REMARK

With the substitutions $x = x_1$ and $y = x_2$, this is the same problem as that given at the top of the preceding page.

EXAMPLE 10**Finding Maximum and Minimum Values**

Find the maximum and minimum values, and the corresponding normalized eigenvectors, of $z = 7x_1^2 + 6x_1x_2 + 7x_2^2$ subject to the constraint $\|\mathbf{x}\|^2 = 1$.

SOLUTION

The quadratic form f can be written using matrix notation as

$$f(x_1, x_2) = \mathbf{x}^T A \mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Verify that the eigenvalues of $A = \begin{bmatrix} 7 & 3 \\ 3 & 7 \end{bmatrix}$ are $\lambda_1 = 10$ and $\lambda_2 = 4$, with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

So, the constrained maximum of 10 occurs when $(x_1, x_2) = \frac{1}{\sqrt{2}}(1, 1) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

and the constrained minimum of 4 occurs when $(x_1, x_2) = \frac{1}{\sqrt{2}}(-1, 1) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

EXAMPLE 11**Using a Change of Variables**

To find the maximum and minimum values of

$$z = 4xy$$

subject to the constraint $9x^2 + 4y^2 = 36$, you cannot use the Constrained Optimization Theorem directly because the constraint is not $\|\mathbf{x}\|^2 = 1$. However, with the change of variables

$$x = 2x' \quad \text{and} \quad y = 3y'$$

the problem becomes finding the maximum and minimum values of

$$z = 24x'y'$$

subject to the constraint $(x')^2 + (y')^2 = 1$. Verify that the maximum value of 12 occurs when $(x', y') = (1/\sqrt{2}, 1/\sqrt{2})$, or $(x, y) = (\sqrt{2}, 3/\sqrt{2})$, and the minimum value of -12 occurs when $(x', y') = (1/\sqrt{2}, -1/\sqrt{2})$, or $(x, y) = (\sqrt{2}, -3/\sqrt{2})$.

7.4 Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Finding Age Distribution Vectors In Exercises 1–6, use the age transition matrix L and age distribution vector $\mathbf{x}_1$ to find the age distribution vectors $\mathbf{x}_2$ and $\mathbf{x}_3$. Then find a stable age distribution vector.

1. $L = \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$

2. $L = \begin{bmatrix} 0 & 4 \\ \frac{1}{16} & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 160 \\ 160 \end{bmatrix}$

3. $L = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix}$

4. $L = \begin{bmatrix} 0 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix}$

5. $L = \begin{bmatrix} 0 & 2 & 2 & 0 \\ \frac{1}{4} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}$

6. $L = \begin{bmatrix} 0 & 6 & 4 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 24 \\ 24 \\ 24 \\ 24 \\ 24 \end{bmatrix}$

7. **Population Growth Model** A population has the characteristics below.

- A total of 75% of the population survives the first year. Of that 75%, 25% survives the second year. The maximum life span is 3 years.
- The average number of offspring for each member of the population is 2 the first year, 4 the second year, and 2 the third year.

The population now consists of 160 members in each of the three age classes. How many members will there be in each age class in 1 year? in 2 years?

8. **Population Growth Model** A population has the characteristics below.

- A total of 80% of the population survives the first year. Of that 80%, 25% survives the second year. The maximum life span is 3 years.
- The average number of offspring for each member of the population is 3 the first year, 6 the second year, and 3 the third year.

The population now consists of 120 members in each of the three age classes. How many members will there be in each age class in 1 year? in 2 years?

9. **Population Growth Model** A population has the characteristics below.

- A total of 60% of the population survives the first year. Of that 60%, 50% survives the second year. The maximum life span is 3 years.
- The average number of offspring for each member of the population is 2 the first year, 5 the second year, and 2 the third year.

The population now consists of 100 members in each of the three age classes. How many members will there be in each age class in 1 year? in 2 years?

10. Find the limit (if it exists) of $A^n \mathbf{x}_1$ as n approaches infinity, where

$$A = \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_1 = \begin{bmatrix} a \\ a \end{bmatrix}.$$

Solving a System of Linear Differential Equations In Exercises 11–20, solve the system of first-order linear differential equations.

11. $y_1' = 2y_1$
 $y_2' = y_2$

12. $y_1' = -5y_1$
 $y_2' = 4y_2$

13. $y_1' = -4y_1$
 $y_2' = -\frac{1}{2}y_2$

14. $y_1' = \frac{1}{2}y_1$
 $y_2' = \frac{1}{8}y_2$

15. $y_1' = -y_1$
 $y_2' = 6y_2$
 $y_3' = y_3$

16. $y_1' = 5y_1$
 $y_2' = -2y_2$
 $y_3' = -3y_3$

17. $y_1' = -0.3y_1$
 $y_2' = 0.4y_2$
 $y_3' = -0.6y_3$

18. $y_1' = -\frac{2}{3}y_1$
 $y_2' = -\frac{3}{5}y_2$
 $y_3' = -8y_3$

19. $y_1' = 7y_1$
 $y_2' = 9y_2$
 $y_3' = -7y_3$
 $y_4' = -9y_4$

20. $y_1' = -0.1y_1$
 $y_2' = -\frac{7}{4}y_2$
 $y_3' = -2\pi y_3$
 $y_4' = \sqrt{5}y_4$

Solving a System of Linear Differential Equations In Exercises 21–28, solve the system of first-order linear differential equations.

21. $y_1' = y_1 - 4y_2$
 $y_2' = 2y_2$

22. $y_1' = y_1 - 4y_2$
 $y_2' = -2y_1 + 8y_2$

23. $y_1' = y_1 + 2y_2$
 $y_2' = 2y_1 + y_2$

24. $y_1' = y_1 - y_2$
 $y_2' = 2y_1 + 4y_2$

25. $y_1' = y_1 - 2y_2 + y_3$
 $y_2' = 2y_2 + 4y_3$
 $y_3' = 3y_3$

26. $y_1' = 2y_1 + y_2 + y_3$
 $y_2' = y_1 + y_2$
 $y_3' = y_1 + y_3$

27. $y_1' = 3y_2 - 5y_3$
 $y_2' = 4y_1 - 4y_2 + 10y_3$
 $y_3' = -4y_3$

28. $y_1' = -2y_1 + y_3$
 $y_2' = 3y_2 + 4y_3$
 $y_3' = y_3$

Writing a System and Verifying the General Solution In Exercises 29–32, write the system of first-order linear differential equations represented by the matrix equation $\mathbf{y}' = A\mathbf{y}$. Then verify the general solution.

29. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $y_1 = C_1 e^t + C_2 t e^t$
 $y_2 = C_2 e^t$

30. $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $y_1 = C_1 e^t \cos t + C_2 e^t \sin t$
 $y_2 = -C_2 e^t \cos t + C_1 e^t \sin t$

31. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 0 \end{bmatrix}$,

$$y_1 = C_1 + C_2 \cos 2t + C_3 \sin 2t$$

$$y_2 = 2C_3 \cos 2t - 2C_2 \sin 2t$$

$$y_3 = -4C_2 \cos 2t - 4C_3 \sin 2t$$

32. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$,

$$y_1 = C_1 e^t + C_2 t e^t + C_3 t^2 e^t$$

$$y_2 = (C_1 + C_2) e^t + (C_2 + 2C_3) t e^t + C_3 t^2 e^t$$

$$y_3 = (C_1 + 2C_2 + 2C_3) e^t + (C_2 + 4C_3) t e^t + C_3 t^2 e^t$$

Finding the Matrix of a Quadratic Form In Exercises 33–38, find the matrix A of the quadratic form associated with the equation.

33. $x^2 + y^2 - 4 = 0$ 34. $x^2 - 4xy + y^2 - 4 = 0$

35. $9x^2 + 10xy - 4y^2 - 36 = 0$

36. $12x^2 - 5xy - x + 2y - 20 = 0$

37. $10xy - 10y^2 + 4x - 48 = 0$

38. $16x^2 - 4xy + 20y^2 - 72 = 0$

Finding the Matrix of a Quadratic Form In Exercises 39–44, find the matrix A of the quadratic form associated with the equation. Then find the eigenvalues of A and an orthogonal matrix P such that $P^T A P$ is diagonal.

39. $2x^2 - 3xy - 2y^2 + 10 = 0$

40. $5x^2 - 2xy + 5y^2 + 10x - 17 = 0$

41. $13x^2 + 6\sqrt{3}xy + 7y^2 - 16 = 0$

42. $3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$

43. $16x^2 - 24xy + 9y^2 - 60x - 80y + 100 = 0$

44. $17x^2 + 32xy - 7y^2 - 75 = 0$

Rotation of a Conic In Exercises 45–52, use the Principal Axes Theorem to perform a rotation of axes to eliminate the xy -term in the quadratic equation. Identify the resulting rotated conic and give its equation in the new coordinate system.

45. $13x^2 - 8xy + 7y^2 - 45 = 0$

46. $x^2 + 4xy + y^2 - 9 = 0$

47. $2x^2 - 4xy + 5y^2 - 36 = 0$

48. $7x^2 + 32xy - 17y^2 - 50 = 0$

49. $2x^2 + 4xy + 2y^2 + 6\sqrt{2}x + 2\sqrt{2}y + 4 = 0$

50. $8x^2 + 8xy + 8y^2 + 10\sqrt{2}x + 26\sqrt{2}y + 31 = 0$

51. $xy + x - 2y + 3 = 0$

52. $5x^2 - 2xy + 5y^2 + 10\sqrt{2}x = 0$

Rotation of a Quadric Surface In Exercises 53–56, find the matrix A of the quadratic form associated with the equation. Then find the equation of the quadric surface in the rotated $x'y'z'$ -system.

53. $3x^2 - 2xy + 3y^2 + 8z^2 - 16 = 0$

54. $2x^2 + 2y^2 + 2z^2 + 2xy + 2xz + 2yz - 1 = 0$

55. $x^2 + 2y^2 + 2z^2 + 2yz - 1 = 0$

56. $x^2 + y^2 + z^2 + 2xy - 8 = 0$

Constrained Optimization In Exercises 57–66, find the maximum and minimum values, and a vector where each occurs, of the quadratic form subject to the constraint.

57. $z = 3x_1^2 + 2x_2^2; \|\mathbf{x}\|^2 = 1$

58. $z = 11x_1^2 + 4x_2^2; \|\mathbf{x}\|^2 = 1$

59. $z = x_1^2 + 12x_2^2; 4x_1^2 + 25x_2^2 = 100$

60. $z = -5x^2 + 9y^2; x^2 + 9y^2 = 9$

61. $z = 5x^2 + 12xy + 5y^2; x^2 + y^2 = 1$

62. $z = 5x_1^2 + 12x_1x_2; \|\mathbf{x}\|^2 = 1$

63. $z = 6x_1x_2; \|\mathbf{x}\|^2 = 1$

64. $z = 9xy; 9x^2 + 16y^2 = 144$

65. $w = x^2 + 3y^2 + z^2 + 2xy + 2xz + 2yz; x^2 + y^2 + z^2 = 1$

66. $w = 2x^2 - y^2 - z^2 + 4xy - 4xz + 8yz; x^2 + y^2 + z^2 = 1$

67. Let P be a 2×2 orthogonal matrix such that $|P| = 1$. Show that there exists a number θ , $0 \leq \theta < 2\pi$, such that

$$P = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

68. CAPSTONE

- Explain how to model population growth using an age transition matrix and an age distribution vector, and how to find a stable age distribution vector.
- Explain how to use a matrix equation to solve a system of first-order linear differential equations.
- Explain how to use the Principal Axes Theorem to perform a rotation of axes for a conic and a quadric surface.
- Explain how to solve a constrained optimization problem.

69. Use your school's library, the Internet, or some other reference source to find real-life applications of constrained optimization.

7 Review Exercises

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Characteristic Equation, Eigenvalues, and Basis In Exercises 1–6, find (a) the characteristic equation of A , (b) the eigenvalues of A , and (c) a basis for the eigenspace corresponding to each eigenvalue.

1. $A = \begin{bmatrix} 2 & 1 \\ 5 & -2 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$

3. $A = \begin{bmatrix} 9 & 4 & -3 \\ -2 & 0 & 6 \\ -1 & -4 & 11 \end{bmatrix}$

4. $A = \begin{bmatrix} -4 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

5. $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix}$

Characteristic Equation, Eigenvalues, and Basis In Exercises 7 and 8, use a software program or a graphing utility to find (a) the characteristic equation of A , (b) the eigenvalues of A , and (c) a basis for the eigenspace corresponding to each eigenvalue.

7. $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

8. $A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Determining Whether a Matrix Is Diagonalizable In Exercises 9–14, determine whether A is diagonalizable. If it is, find a nonsingular matrix P such that $P^{-1}AP$ is diagonal.

9. $A = \begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}$

10. $A = \begin{bmatrix} \frac{1}{6} & \frac{1}{4} \\ \frac{2}{3} & 0 \end{bmatrix}$

11. $A = \begin{bmatrix} -2 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

12. $A = \begin{bmatrix} 3 & -2 & 2 \\ -2 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

13. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

14. $A = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -1 & 1 & 0 \end{bmatrix}$

15. For what value(s) of a does the matrix

$$A = \begin{bmatrix} 0 & 1 \\ a & 1 \end{bmatrix}$$

have the characteristics below?

- (a) A has an eigenvalue of multiplicity 2.
- (b) A has -1 and 2 as eigenvalues.
- (c) A has real eigenvalues.

16. Show that if $0 < \theta < \pi$, then the transformation for a counterclockwise rotation through an angle θ has no real eigenvalues.

Writing In Exercises 17–20, explain why the matrix is not diagonalizable.

17. $A = \begin{bmatrix} 0 & 9 \\ 0 & 0 \end{bmatrix}$

18. $A = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$

19. $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

20. $A = \begin{bmatrix} -2 & 3 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & -2 \end{bmatrix}$

Determining Whether Two Matrices Are Similar In Exercises 21–24, determine whether the matrices are similar. If they are, find a matrix P such that $A = P^{-1}BP$.

21. $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

22. $A = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$

23. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

24. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & -3 & -3 \\ 3 & -5 & -3 \\ -3 & 3 & 1 \end{bmatrix}$

Determining Symmetric and Orthogonal Matrices In Exercises 25–32, determine whether the matrix is symmetric, orthogonal, both, or neither.

25. $A = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

26. $A = \begin{bmatrix} \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} \end{bmatrix}$

27. $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

28. $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

29. $A = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$

30. $A = \begin{bmatrix} \frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ -\frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$

31. $A = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$

32. $A = \begin{bmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} & \frac{2\sqrt{3}}{3} & 0 \\ \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3} \end{bmatrix}$

Eigenvectors of a Symmetric Matrix In Exercises 33–36, show that any two eigenvectors of the symmetric matrix corresponding to distinct eigenvalues are orthogonal.

$$33. \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

$$34. \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

$$35. \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$36. \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Orthogonally Diagonalizable Matrices In Exercises 37–40, determine whether the matrix is orthogonally diagonalizable.

$$37. \begin{bmatrix} -3 & -1 \\ -1 & -2 \end{bmatrix}$$

$$38. \begin{bmatrix} -4 & 1 \\ -1 & 3 \end{bmatrix}$$

$$39. \begin{bmatrix} 4 & 1 & 2 \\ 0 & -1 & 0 \\ 2 & 1 & -5 \end{bmatrix}$$

$$40. \begin{bmatrix} 5 & 4 & -1 \\ 4 & 1 & 3 \\ -1 & 3 & -2 \end{bmatrix}$$

Orthogonal Diagonalization In Exercises 41–46, find a matrix P that orthogonally diagonalizes A . Verify that P^TAP gives the correct diagonal form.

$$41. A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

$$42. A = \begin{bmatrix} 8 & 15 \\ 15 & -8 \end{bmatrix}$$

$$43. A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$44. A = \begin{bmatrix} 3 & 0 & -3 \\ 0 & -3 & 0 \\ -3 & 0 & 3 \end{bmatrix}$$

$$45. A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

$$46. A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Steady State Probability Vector In Exercises 47–54, find the steady state probability vector for the matrix. An eigenvector $\mathbf{v}$ of an $n \times n$ matrix A is a steady state probability vector when $A\mathbf{v} = \mathbf{v}$ and the components of $\mathbf{v}$ sum to 1.

$$47. A = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix}$$

$$48. A = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$$

$$49. A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

$$50. A = \begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{bmatrix}$$

$$51. A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$52. A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix}$$

$$53. A = \begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.1 & 0.2 & 0.8 \end{bmatrix}$$

$$54. A = \begin{bmatrix} 0.3 & 0.1 & 0.4 \\ 0.2 & 0.4 & 0.0 \\ 0.5 & 0.5 & 0.6 \end{bmatrix}$$

55. Proof Prove that if A is an $n \times n$ symmetric matrix, then P^TAP is symmetric for any $n \times n$ matrix P .

56. Show that the characteristic polynomial of

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix}$$

is $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0$. A is called the **companion matrix** of the polynomial p .

Finding the Companion Matrix and Eigenvalues In Exercises 57 and 58, use the result of Exercise 56 to find the companion matrix A of the polynomial and find the eigenvalues of A .

$$57. p(\lambda) = 4\lambda^2 - 9\lambda$$

$$58. p(\lambda) = 2\lambda^3 - 7\lambda^2 - 120\lambda + 189$$

59. The characteristic equation of

$$A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$$

is $\lambda^2 - 10\lambda + 24 = 0$. Using $A^2 - 10A + 24I_2 = O$, you can find powers of A by the process below.

$$A^2 = 10A - 24I_2, \quad A^3 = 10A^2 - 24A, \\ A^4 = 10A^3 - 24A^2, \dots$$

Use this process to find the matrices A^2 , A^3 , and A^4 .

60. Repeat Exercise 59 for the matrix

$$A = \begin{bmatrix} 9 & 4 & -3 \\ -2 & 0 & 6 \\ -1 & -4 & 11 \end{bmatrix}$$

61. Proof Let A be an $n \times n$ matrix.

(a) Prove or disprove that an eigenvector of A is also an eigenvector of A^2 .

(b) Prove or disprove that an eigenvector of A^2 is also an eigenvector of A .

62. Proof Let A be an $n \times n$ matrix. Prove that if $A\mathbf{x} = \lambda\mathbf{x}$, then $\mathbf{x}$ is an eigenvector of $(A + cI)$, where λ and c are scalars. What is the corresponding eigenvalue?

63. Proof Let A and B be $n \times n$ matrices. Prove that if A is nonsingular, then AB is similar to BA .

64. (a) Find a symmetric matrix B such that $B^2 = A$ for

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(b) Generalize the result of part (a) by proving that if A is an $n \times n$ symmetric matrix with positive eigenvalues, then there exists a symmetric matrix B such that $B^2 = A$.

65. Determine all $n \times n$ symmetric matrices that have 0 as their only eigenvalue.

66. Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal for the matrix

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

67. **Writing** Let A be an $n \times n$ idempotent matrix (that is, $A^2 = A$). Describe the eigenvalues of A .

68. **Writing** The matrix below has an eigenvalue $\lambda = 2$ of multiplicity 4.

$$A = \begin{bmatrix} 2 & a & 0 & 0 \\ 0 & 2 & b & 0 \\ 0 & 0 & 2 & c \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

- (a) Under what conditions is A diagonalizable?
 (b) Under what conditions does the eigenspace of $\lambda = 2$ have dimension 1? 2? 3?

True or False? In Exercises 69 and 70, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

69. (a) An eigenvector of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ in R^n such that $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$.
 (b) Similar matrices may or may not have the same eigenvalues.
 (c) To diagonalize a square matrix A , you need to find an invertible matrix P such that $P^{-1}AP$ is diagonal.
70. (a) An eigenvalue of a matrix A is a scalar λ such that $\det(\lambda I - A) = 0$.
 (b) An eigenvector may be the zero vector $\mathbf{0}$.
 (c) A matrix A is orthogonally diagonalizable when there exists an orthogonal matrix P such that $P^{-1}AP = D$ is diagonal.

Finding Age Distribution Vectors In Exercises 71–74, use the age transition matrix L and the age distribution vector $\mathbf{x}_1$ to find the age distribution vectors $\mathbf{x}_2$ and $\mathbf{x}_3$. Then find a stable age distribution vector.

71. $L = \begin{bmatrix} 0 & 1 \\ \frac{1}{4} & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$

72. $L = \begin{bmatrix} 0 & 1 \\ \frac{3}{4} & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 32 \\ 32 \end{bmatrix}$

73. $L = \begin{bmatrix} 0 & 3 & 12 \\ 1 & 0 & 0 \\ 0 & \frac{1}{6} & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 300 \\ 300 \\ 300 \end{bmatrix}$

74. $L = \begin{bmatrix} 0 & 2 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 240 \\ 240 \\ 240 \end{bmatrix}$

75. **Population Growth Model** A population has the characteristics below.

- (a) A total of 90% of the population survives the first year. Of that 90%, 75% survives the second year. The maximum life span is 3 years.
 (b) The average number of offspring for each member of the population is 4 the first year, 6 the second year, and 2 the third year.

The population now consists of 120 members in each of the three age classes. How many members will there be in each age class in 1 year? in 2 years?

76. **Population Growth Model** A population has the characteristics below.

- (a) A total of 75% of the population survives the first year. Of that 75%, 60% survives the second year. The maximum life span is 3 years.
 (b) The average number of offspring for each member of the population is 4 the first year, 8 the second year, and 2 the third year.

The population now consists of 120 members in each of the three age classes. How many members will there be in each age class in 1 year? in 2 years?

Solving a System of Linear Differential Equations In Exercises 77–80, solve the system of first-order linear differential equations.

77. $y_1' = 3y_1$
 $y_2' = y_1 - y_2$

78. $y_1' = y_2$
 $y_2' = y_1$

79. $y_1' = 3y_1$
 $y_2' = 8y_2$
 $y_3' = -8y_3$

80. $y_1' = 6y_1 - y_2 + 2y_3$
 $y_2' = 3y_2 - y_3$
 $y_3' = y_3$

Rotation of a Conic In Exercises 81–84, (a) find the matrix A of the quadratic form associated with the equation, (b) find an orthogonal matrix P such that P^TAP is diagonal, (c) use the Principal Axes Theorem to perform a rotation of axes to eliminate the xy -term in the quadratic equation, and (d) sketch the graph of each equation.

81. $x^2 + 3xy + y^2 - 3 = 0$

82. $x^2 - \sqrt{3}xy + 2y^2 - 10 = 0$

83. $xy - 2 = 0$

84. $9x^2 - 24xy + 16y^2 - 400x - 300y = 0$

Constrained Optimization In Exercises 85–88, find the maximum and minimum values, and a vector where each occurs, of the quadratic form subject to the constraint.

85. $z = x^2 - y^2; x^2 + y^2 = 1$

86. $z = x_1x_2; 25x_1^2 + 4x_2^2 = 100$

87. $z = 15x_1^2 - 4x_1x_2 + 15x_2^2; \|\mathbf{x}\|^2 = 1$

88. $z = -11x^2 + 10xy - 11y^2; x^2 + y^2 = 1$

7 Projects



1 Population Growth and Dynamical Systems (I)

Systems of differential equations often arise in biological applications of population growth of various species of animals. These equations are called **dynamical systems** because they describe the changes in a system as functions of time. Assume that a biologist studies the populations of predator sharks $y_1(t)$ and their small fish prey $y_2(t)$ over time t . One model for the relative growths of these populations is

$$y_1'(t) = ay_1(t) + by_2(t) \quad \text{Predator}$$

$$y_2'(t) = cy_1(t) + dy_2(t) \quad \text{Prey}$$

where a, b, c , and d are constants. The constants a and d are positive, reflecting the growth rates of the species. In a predator-prey relationship, $b > 0$ and $c < 0$ because an increase in prey fish y_2 would cause an increase in predator sharks y_1 , whereas an increase in y_1 would cause a decrease in y_2 .

The system of linear differential equations below models the populations of sharks $y_1(t)$ and prey fish $y_2(t)$ with the populations at time $t = 0$.

$$y_1'(t) = 0.5y_1(t) + 0.6y_2(t) \quad y_1(0) = 36$$

$$y_2'(t) = -0.4y_1(t) + 3.0y_2(t) \quad y_2(0) = 121$$

1. Use the diagonalization techniques of this chapter to find the populations $y_1(t)$ and $y_2(t)$ at any time $t > 0$.
2. Interpret the solutions in terms of the long-term population trends for the two species. Does one species ultimately disappear? Why or why not?
3. Graph the solutions $y_1(t)$ and $y_2(t)$ over the domain $0 \leq t \leq 3$.
4. Explain why the quotient $y_2(t)/y_1(t)$ approaches a limit as t increases.

2 The Fibonacci Sequence

The **Fibonacci sequence** is named after the Italian mathematician Leonard Fibonacci of Pisa (1170–1250). To form this sequence, define the first two terms as $x_1 = 1$ and $x_2 = 1$, and then define the n th term as the sum of its two immediate predecessors. That is, $x_n = x_{n-1} + x_{n-2}$. So, the third term is $2 = 1 + 1$, the fourth term is $3 = 2 + 1$, and so on. The formula $x_n = x_{n-1} + x_{n-2}$ is called *recursive* because the first $n - 1$ terms must be calculated before the n th term can be calculated. In this project, you will use eigenvalues and diagonalization to derive an explicit formula for the n th term of the Fibonacci sequence.

1. Calculate the first 12 terms of the Fibonacci sequence.
2. Explain how the matrix identity $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ x_{n-2} \end{bmatrix} = \begin{bmatrix} x_{n-1} + x_{n-2} \\ x_{n-1} \end{bmatrix}$ can be used to generate the Fibonacci sequence recursively.
3. Starting with $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, show that $A^{n-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$, where $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.
4. Find a matrix P that diagonalizes A .
5. Derive an explicit formula for the n th term of the Fibonacci sequence. Use this formula to calculate x_1, x_2 , and x_3 .
6. Determine the limit of x_n/x_{n-1} as n approaches infinity. Do you recognize this number?

REMARK

You can learn more about dynamical systems and population modeling in most books on differential equations. You can learn more about Fibonacci numbers in most books on number theory. You might find it interesting to look at the *Fibonacci Quarterly*, the official journal of the Fibonacci Association.

6 and 7 Cumulative Test

See CalcChat.com for worked-out solutions to odd-numbered exercises.



Take this test to review the material in Chapters 6 and 7. After you are finished, check your work against the answers in the back of the book.

In Exercises 1 and 2, determine whether the function is a linear transformation.

1. $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $T(x, y, z) = (2x, x + y)$ 2. $T: M_{2,2} \rightarrow \mathbb{R}$, $T(A) = |A + A^T|$

3. Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be the linear transformation defined by $T(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 3 & 0 & 2 \end{bmatrix}.$$

Find the dimensions of $\mathbb{R}^n$ and $\mathbb{R}^m$.

4. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the linear transformation defined by $T(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} -2 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Find (a) $T(2, -1)$ and (b) the preimage of $(-6, 3, 0)$.

5. Find the kernel of the linear transformation

$$T: \mathbb{R}^4 \rightarrow \mathbb{R}^4$$
, $T(x_1, x_2, x_3, x_4) = (x_1 - x_2, x_2 - x_1, 0, x_3 + x_4)$.

6. Let $T: \mathbb{R}^4 \rightarrow \mathbb{R}^2$ be the linear transformation defined by $T(\mathbf{v}) = A\mathbf{v}$, where

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}.$$

Find a basis for (a) the kernel of T and (b) the range of T . (c) Determine the rank and nullity of T .

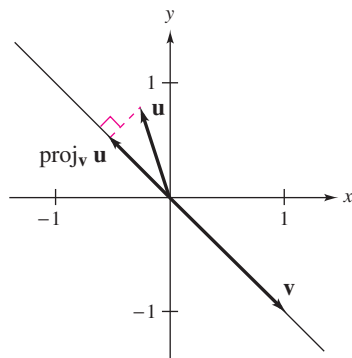


Figure for 11

In Exercises 7–10, find the standard matrix for the linear transformation T .

7. $T(x, y) = (3x + 2y, 2y - x)$ 8. $T(x, y, z) = (x + y, y + z, x - z)$

9. $T(x, y, z) = (3z - 2y, 4x + 11z)$ 10. $T(x_1, x_2, x_3) = (0, 0, 0)$

11. Find the standard matrix A for the linear transformation $\text{proj}_{\mathbf{v}}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that projects an arbitrary vector $\mathbf{u}$ onto the vector $\mathbf{v} = [1 \ -1]^T$, as shown in the figure. Use this matrix to find the images of the vectors $(1, 1)$ and $(-2, 2)$.

12. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation defined by a counterclockwise rotation of 30° in $\mathbb{R}^2$.

(a) Find the standard matrix A for the linear transformation.

(b) Use A to find the image of the vector $\mathbf{v} = (1, 2)$.

(c) Sketch the graph of $\mathbf{v}$ and its image.

In Exercises 13 and 14, find the standard matrices for $T = T_2 \circ T_1$ and $T' = T_1 \circ T_2$.

13. $T_1: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T_1(x, y) = (x - 2y, 2x + 3y)$

$$T_2: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$
, $T_2(x, y) = (2x, x - y)$

14. $T_1: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $T_1(x, y, z) = (x + 2y, y - z, -2x + y + 2z)$

$$T_2: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$
, $T_2(x, y, z) = (y + z, x + z, 2y - 2z)$

15. Find the inverse of the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x, y) = (x - y, 2x + y)$. Verify that $(T^{-1} \circ T)(3, -2) = (3, -2)$.

16. Determine whether the linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_1 + x_3)$ is invertible. If it is, find its inverse.

17. Find the matrix of the linear transformation $T(x, y) = (y, 2x, x + y)$ relative to the bases $B = \{(1, 1), (1, 0)\}$ for R^2 and $B' = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ for R^3 . Use this matrix to find the image of the vector $(0, 1)$.
18. Let $B = \{(1, 0), (0, 1)\}$ and $B' = \{(1, 1), (1, 2)\}$ be bases for R^2 .
- Find the matrix A of $T: R^2 \rightarrow R^2$, $T(x, y) = (x - 2y, x + 4y)$, relative to the basis B .
 - Find the transition matrix P from B' to B .
 - Find the matrix A' of T relative to the basis B' .
 - Find $[T(\mathbf{v})]_{B'}$ when $[\mathbf{v}]_{B'} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.
 - Verify your answer in part (d) by finding $[\mathbf{v}]_B$ and $[T(\mathbf{v})]_B$.

In Exercises 19–22, find the eigenvalues and the corresponding eigenvectors of the matrix.

19. $\begin{bmatrix} 7 & 2 \\ -2 & 3 \end{bmatrix}$

20. $\begin{bmatrix} -15 & -5 \\ 0 & 5 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & -3 & -1 \end{bmatrix}$

22. $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

In Exercises 23 and 24, find a nonsingular matrix P such that $P^{-1}AP$ is diagonal.

23. $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

24. $A = \begin{bmatrix} 0 & -3 & 5 \\ -4 & 4 & -10 \\ 0 & 0 & 4 \end{bmatrix}$

25. Find a basis B for R^3 such that the matrix for the linear transformation $T: R^3 \rightarrow R^3$, $T(x, y, z) = (2x - 2z, 2y - 2z, 3x - 3z)$, relative to B is diagonal.
26. Find an orthogonal matrix P such that P^TAP diagonalizes the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.
27. Use the Gram-Schmidt orthonormalization process to find an orthogonal matrix P such that P^TAP diagonalizes the symmetric matrix $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$.

28. Solve the system of differential equations.

$$\begin{aligned} y_1' &= y_1 \\ y_2' &= 9y_2 \end{aligned}$$

29. Find the matrix of the quadratic form associated with the quadratic equation $3x^2 - 16xy + 3y^2 - 13 = 0$.

30. A population has the following characteristics.

- A total of 80% of the population survives the first year. Of that 80%, 40% survives the second year. The maximum life span is 3 years.
- The average number of offspring for each member of the population is 3 the first year, 6 the second year, and 3 the third year.

The population now consists of 150 members in each of the three age classes. How many members will there be in each age class in 1 year? in 2 years?

31. Define an *orthogonal matrix*.

32. Prove that if A is similar to B and A is diagonalizable, then B is diagonalizable.